Effect of Medium Attenuation on the Asymptotic Eigenvalues of Noise Covariance Matrices

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***Abstract*—Covariance matrices of noise models are used in signal and array processing to study the effect of various noise fields and array configurations on signals and their detectability. Here, the asymptotic eigenvalues of noise covariance matrices in 2-D and 3-D attenuating media are derived. The asymptotic eigenvalues are given by a continuous function, which is the Fourier transform of the infinite sequence formed by sampling the spatial coherence function. The presence of attenuation decreases the value of the large eigenvalues and raises the value of the smaller eigenvalues (compared to the attenuation free case). The eigenvalue density of the sample covariance matrix also shows variation in shape depending on the attenuation, which potentially could be used to retrieve medium attenuation properties from observations of noise.**

***Index Terms*—Attenuating media, covariance matrix, eigenvalues, spatial coherence function.**

I. Introduction

COVARIANCE matrices (CMs) play a central role in several applications such as direction of arrival estimation [1], [2], signal detection from limited data [3]–[5], channel estimation in wireless communications [6], [7] and ambient noise processing [8]–[10]. Estimating the CM or its eigenvalues from finite observations [11], [12] also plays an important role in these applications. The presence of structure in the CM (e.g., a Hermitian Toeplitz matrix [13]) makes it possible to gain analytical insights into the asymptotic behavior of the eigenvalues (dx.doi.org/10.1109/MSP. 2012.2207490).

Here, we are concerned with the spatial CMs from measurements of noise on a uniform line array (ULA). We derive the asymptotic eigenvalues for CMs in 2-D and 3-D media with attenuation, and demonstrate that the attenuation has a significant effect on the eigenvalues of the CM. We futher demonstrate using a random matrix theory based approximation [14], that the eigenvalue density of the sample covariance matrix (SCM) is also affected by the attenuation, which is of interest in the development of signal processing algorithms based on the physics of propagation in a given medium.

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II. Background

A. The Spatial Coherence Function

Consider two sensors Δx apart, and the signals recorded on them, $\psi_1(f)$ and $\psi_2(f)$ at frequency f. The respective power spectral densities are $P_{11}(f) = \langle\psi_1(f)\psi_1^\dagger(f)\rangle$ and $P_{22}(f) = \langle\psi_2(f)\psi_2^\dagger(f)\rangle$ where † denotes the complex conjugate and $\langle\cdot\rangle$ the ensemble average. The spatial coherence function (SCF) between these two sensors is defined as the normalized cross-spectral density (henceforth, the dependence on f is dropped),

$$\Gamma = \frac{P_{12}}{\sqrt{P_{11}P_{22}}}. \tag{1}$$

The functional forms of Γ are well known for uncorrelated and uniformly distributed sources in the medium. In 2-D and 3-D media without attenuation, the respective SCFs are [15]

$$\Gamma_0^{\mathrm{2D}}(\beta) = \mathrm{J}_0(2\pi\beta) \tag{2}$$

$$\Gamma_0^{\mathrm{3D}}(\beta) = \mathrm{sinc}(2\beta) \tag{3}$$

where $\mathrm{sinc}(z) = \sin(\pi z)/\pi z$, $\beta = f\Delta x/c \geq 0$ is the spacing to wavelength ratio, c is the phase speed and the subscript 0 indicates no attenuation. The SCFs in (2) and (3) are not dependent on the individual sensor locations, but only on their separation distance.

The effect of medium attenuation (assumed homogeneous) is introduced using a term that exponentially decays with the distance of separation between the sensors (which manifests in β) [16]–[19] and the SCF is

$$\Gamma_\delta(\beta) = e^{-2\pi\beta\delta}\Gamma_0(\beta) \tag{4}$$

where δ is the *loss tangent*.

B. The Spatial Covariance Matrix

For a ULA of N sensors with a uniform spacing Δx, the SCF of the noise field between the ith and the jth sensors is $\Gamma_\delta(\beta|i-j|)$ where β now is redefined as the spacing to wavelength ratio for adjacent sensors. The normalized CM $\mathbf{\Sigma}_\delta$ (or the normalized cross-spectral density matrix) of the frequency domain observations at frequency f is then related to Γ_δ as

$$\mathbf{\Sigma}_\delta(i,j) = \Gamma_\delta(\beta|i-j|). \tag{5}$$

$\Gamma_\delta(\beta|i-j|)$, with constant β can be considered to be sampled from the continuous function $\Gamma_\delta(\beta|x|)$ at integer values of x. $\Gamma_\delta(\beta|x|)$ is used in Section III to derive the asymptotic eigenvalues of $\mathbf{\Sigma}_\delta$.

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The spatial CM is an important quantity in analyzing noise data, as the time domain cross-correlation between any two pairs of sensors can be obtained simply by taking the inverse Fourier transform of the corresponding element (as a function of frequency). The framework also allows one to use eigenanalysis techniques such as principal components analysis and RMT.

The CM in (5) is a Hermitian Toeplitz matrix with real and non-negative eigenvalues. The asymptotic eigenvalues of $\mathbf{\Sigma}_\delta$ as $N \to \infty$ are related to the Fourier transform of the underlying infinite sequence $\Gamma_\delta(\beta|n|),\ n \in \mathbb{Z}$ in the Toeplitz matrix (sampled from $\Gamma_\delta(\beta|x|)$ with a sampling interval of 1), if the sequence is absolutely summable [13]. The property of absolute summability of $\Gamma_\delta(\beta|n|)$ guarantees the existence of its Fourier transform $\phi_\delta(\kappa)$ and the absolute convergence of the error between the samples of $\phi_\delta(\kappa)$ and the eigenvalues of $\mathbf{\Sigma}_\delta$ as $N \to \infty$.

III. Asymptotic Eigenvalues of the CM

For a general CM with a corresponding Γ_δ, which may not be absolutely summable, the matrix based approach in [8] may be used to show a weaker convergence of the error term. The Fourier transform of Γ_δ is

$$\phi_\delta(\kappa) = \sum_{n=-\infty}^{\infty} \Gamma_\delta(\beta|n|) e^{-\imath 2\pi\kappa n} \tag{6}$$

where $\kappa \in [-1/2, 1/2)$ is the normalized spatial frequency (cycles/sample). $\phi_\delta(\kappa)$ is a continuous function describing the asymptotic eigenvalues of $\mathbf{\Sigma}_\delta$. Using the continuous Fourier transform (CFT) $\mathcal{G}_\delta(\gamma)$ of the SCF,

$$\mathcal{G}_\delta(\gamma) = \int_{-\infty}^{\infty} \Gamma_\delta(\beta|x|) e^{-\imath 2\pi\gamma x}\, \mathrm{d}x, \tag{7}$$

where γ is the ordinary spatial frequency, (6) is written as

$$\phi_\delta(\kappa) = \sum_{n=-\infty}^{\infty} \mathcal{G}_\delta(\kappa - n). \tag{8}$$

Since Γ_δ is real and even, $\mathcal{G}_\delta$ and ϕ_δ are also real and even. For the attenuation free case, $\Gamma_0^{2\mathrm{D}}$ and $\Gamma_0^{3\mathrm{D}}$ are bandlimited functions, hence only the $n = 0$ term contributes in (8) for $\beta \leq 1/2$ (i.e., no spatial aliasing).

In the following section, we obtain the asymptotic eigenvalues of the CM in attenuating media for the 2-D and 3-D noise fields i.e., when the SCF is given by (4) and also establish the convergence of empirical results for 2-D media without attenuation.

A. 2-D Medium With Attenuation

The infinite sequence $\Gamma_\delta^{2\mathrm{D}}(\beta|n|)$ from (4), is absolutely summable because of the presence of an attenuation term. Noting that $|\mathrm{J}_0(x)| \leq 1$, we have $\forall \delta > 0$:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |\Gamma_\delta^{2\mathrm{D}}(\beta|n|)| &= \sum_{n=-\infty}^{\infty} |e^{-2\pi\beta\delta|n|} \mathrm{J}_0(2\pi\beta n)| \\ &\leq \sum_{n=-\infty}^{\infty} e^{-2\pi\beta\delta|n|} = \frac{e^{2\pi\beta\delta}+1}{e^{2\pi\beta\delta}-1} < \infty\end{aligned}$$

The CFT of the SCF $\Gamma_\delta^{2\mathrm{D}}$ is evaluated by using the integral representation of the Bessel function:

$$\begin{aligned}\mathcal{G}_\delta^{2\mathrm{D}}(\gamma) &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-\imath 2\pi\gamma x} e^{-2\pi\beta\delta|x|} \int_0^{\pi} e^{\imath 2\pi\beta x\cos(\theta)}\, \mathrm{d}\theta\, \mathrm{d}x \\ &= \Re\left[\frac{1}{\pi^2}\int_0^{\pi} \frac{\mathrm{d}\theta}{\beta\delta + \imath\gamma - \imath\beta\cos(\theta)}\right] \\ &= \frac{1}{\pi}\Re\left[\frac{1}{\sqrt{(\beta\delta+\imath\gamma)^2+\beta^2}}\right]\end{aligned} \tag{9}$$

(the second step follows from the fact that the imaginary part of the integrand is odd, and the last step is obtained using the substitution $u = \tan\theta/2$). $\phi_\delta^{2\mathrm{D}}(\kappa)$ is then obtained using (8) and (9).

When $\delta = 0$ (i.e., no attenuation), $\mathcal{G}_\delta^{2\mathrm{D}}$ in (9) reduces to the CFT of $\Gamma_0^{2\mathrm{D}}$ in (2) and $\phi_0^{2\mathrm{D}}(\kappa)$ for $\beta \leq 1/2$ [only the $n = 0$ term contributes in (8)] is [10]

$$\phi_0^{2\mathrm{D}}(\kappa) = \frac{\mathrm{rect}\left(\frac{\kappa}{2\beta}\right)}{\pi\sqrt{\beta^2-\kappa^2}} \tag{10}$$

where $\mathrm{rect}(x) = 1\ \forall\ |x| \leq 1/2$ and 0 otherwise. The convergence of $\Gamma_0^{2\mathrm{D}}(\beta|n|)$ is shown in the appendix.

B. 3-D Medium With Attenuation

Just as in Section III-A, $\Gamma_\delta^{3\mathrm{D}}(\beta|n|)$ from (4) can also be shown to be absolutely summable. Its CFT is evaluated by convolving the Fourier transforms of each of the terms in the expression as:

$$\begin{aligned}\mathcal{G}_\delta^{3\mathrm{D}}(\gamma) &= \mathscr{F}\left[e^{-2\pi\beta\delta|x|}\right] \star \mathscr{F}[\mathrm{sinc}(2\beta x)] \\ &= \left[\frac{1}{\pi}\frac{\beta\delta}{\beta^2\delta^2+\gamma^2}\right] \star \left[\frac{1}{2\beta}\mathrm{rect}\left(\frac{\gamma}{2\beta}\right)\right]\end{aligned}$$

where $\mathscr{F}$ denotes the CFT. Then, it follows that

$$\begin{aligned}\mathcal{G}_\delta^{3\mathrm{D}}(\gamma) &= \frac{\delta}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{rect}\left(\frac{\gamma'}{2\beta}\right)}{\beta^2\delta^2+(\gamma-\gamma')^2}\, \mathrm{d}\gamma' \\ &= \frac{1}{2\pi\beta}\left[\tan^{-1}\left(\frac{\gamma+\beta}{\beta\delta}\right) - \tan^{-1}\left(\frac{\gamma-\beta}{\beta\delta}\right)\right]\end{aligned} \tag{11}$$

Finally, $\phi_\delta^{3\mathrm{D}}(\kappa)$ is obtained using (8) and (11).

When $\delta = 0$ (i.e., no attenuation), $\Gamma_\delta^{3\mathrm{D}}$ in (11) reduces to the CFT of $\Gamma_0^{3\mathrm{D}}$ in (3) and $\phi_0^{3\mathrm{D}}(\kappa)$ for $\beta \leq 1/2$ [only the $n = 0$ term contributes in (8)] is [8]

$$\phi_0^{3\mathrm{D}}(\kappa) = \frac{1}{2\beta}\mathrm{rect}\left(\frac{\kappa}{2\beta}\right) \tag{12}$$

IV. Discussion

A. Effect of Attenuation on the Bandwidth and the Eigenvalues

The presence of attenuation ($\delta \neq 0$) makes the SCF an infinite bandwidth function (bandwidth here refers to the support in the Fourier domain). Hence in general, simplified expressions for $\phi_\delta(\kappa)$ cannot be obtained as a function of δ, and must be computed using the sum (8). The effect of the attenuation term on the SCF is shown in Fig. 1(a) for $\Gamma_\delta^{2\mathrm{D}}$ for $\delta = 0$ (no attenuation), 0.1, and 1 [Fig. 1(d) for $\Gamma_\delta^{3\mathrm{D}}$]. The corresponding eigenvalues

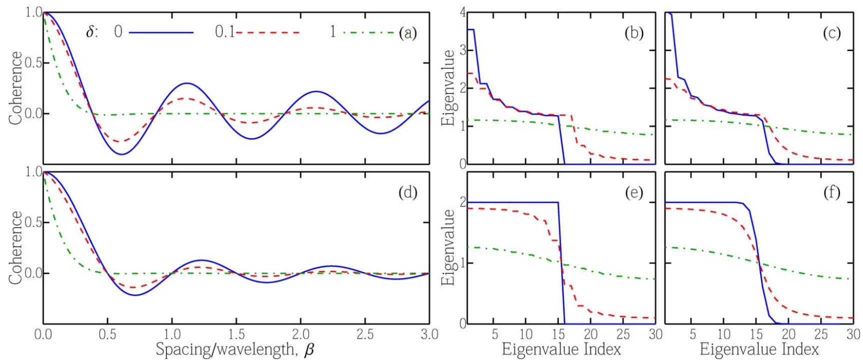

Fig. 1. (a) SCF $\Gamma_\delta^{2\mathrm{D}}(\beta)$ for $\delta = 0$ (solid), 0.1 (dashed) and 1 (dot-dashed). (b) Approximate eigenvalues from sampling the Fourier transform $\phi_\delta^{2\mathrm{D}}(\kappa)$ of the SCF at $N = 30$ equispaced samples in $[-1/2, 1/2)$ for $\beta = 1/4$ and (c) numerical eigenvalues of the spatial CM $\mathbf{\Sigma}_\delta$ for $N = 30$ sensors for the same values of δ and β. (d)–(f) Same as in (a)–(c) for 3-D media.

of $\mathbf{\Sigma}_\delta$ for $N = 30$ sensors, with $\beta = 1/4$ [Fig. 1(c), (f)] are approximated well by N equispaced samples [Fig. 1(b), (e)] from $\phi_\delta(\kappa)$.

The broadening of the support (or bandwidth) due to an increase in attenuation visibly lowers the larger eigenvalues of the CM and raises the smaller eigenvalues [for example, in Fig. 1(c), $\lambda_{1,2} \approx 4$ for $\delta = 0$ (no attenuation) and ≈ 2.4 for $\delta = 0.1$]. In addition, the transition region (near index 15) between the large eigenvalues and the small eigenvalues, which is sharp for no attenuation ($\delta = 0$), spreads out as δ increases. The broadening of the transition region is more apparent in the 3-D case [Fig. 1(e), (f)]. As a result, the smaller eigenvalues which were zero in the attenuation free case [8], [10], no longer are, and this increases the rank of the CM. For $\delta = 1$, i.e., high attenuation, the eigenvalues are all close to 1 and become exactly 1 when $\delta \to \infty$.

B. Eigenvalue Density of the Sample Covariance Matrix

In practice, the CM is often unknown and one uses the SCM estimated from M observations of the data. The SCM $\widehat{\mathbf{\Sigma}}_\delta$ is modeled as

$$\widehat{\mathbf{\Sigma}}_\delta = \frac{1}{M}\mathbf{\Sigma}_\delta^{1/2}\mathbf{X}\mathbf{X}^H\mathbf{\Sigma}_\delta^{1/2} \tag{13}$$

where $\mathbf{X}$ is an $N \times M$ random matrix whose entries are drawn from $\mathcal{CN}(0,1)$, and $\mathbf{\Sigma}_\delta^{1/2}$ is a non-negative definite square root of the true CM, $\mathbf{\Sigma}_\delta$.

Obtaining the SCM eigenvalue density from $\phi_\delta(\kappa)$ by naïvely using Stieltjes transforms is non-trivial due to the eigenvalue structure in $\phi_\delta(\kappa)$. Using a computationally efficient approach [14] using the polynomial method [20], it is possible to approximate the eigenvalue density of the SCM in the attenuated case (if some of the eigenvalues are zero, only the non-zero eigenvalues used and scaled appropriately).

The N equispaced samples from $\phi_\delta(\kappa)$ for $\kappa \in [-1/2, 1/2)$, $\{\lambda_1, \ldots, \lambda_N\}$ (sorted largest first) are divided into three sets:

$$\begin{aligned}\Lambda_\delta^{\mathrm{dist}} &= \{\lambda_i | \lambda_i > \lambda_N(1+\sqrt{\nu})^2\}\\ \Lambda_\delta^{\mathrm{mid}} &= \{\lambda_i | \lambda_N(1+\sqrt{\nu}) < \lambda_i \le \lambda_N(1+\sqrt{\nu})^2\}\\ \Lambda_\delta^{\mathrm{low}} &= \{\lambda_i | \lambda_i \le \lambda_N(1+\sqrt{\nu})\}\end{aligned} \tag{14}$$

where $\nu = N/M$. $\Lambda_\delta^{\mathrm{dist}}$ denotes the set of large eigenvalues that are "well separated" from the rest and each have a distinct contribution to the density. Sets $\Lambda_\delta^{\mathrm{mid}}$ and $\Lambda_\delta^{\mathrm{low}}$ denote eigenvalues with similar spreading behavior and can each be replaced by a single representative eigenvalue weighted appropriately [21]. Accordingly, the Stieltjes transform of the SCM eigenvalue density can then be written as

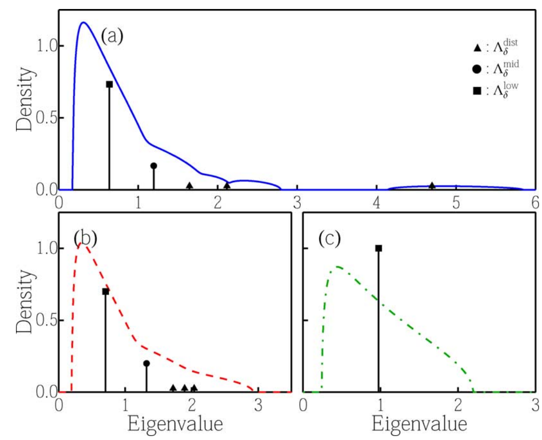

Fig. 2. Approximate eigenvalue density of the SCM for the 2-D attenuated case when the number of sensors $N = 30$, observations $M = 120$, spacing/wavelength $\beta = 1/2$ and attenuation $\delta = \mathrm{(a)}\ 0,\ \mathrm{(b)}\ 0.1$ and (c) 1. The markers indicate the representative eigenvalues and their weights used in the Stieltjes transform for the different groups in (14).

$$\hat{s}(z) = \sum_{\lambda_i \in \Lambda_\delta^{\mathrm{dist}}} \frac{\frac{1}{N}}{\lambda_i\chi - z} + \sum_{\lambda_i \in \Lambda_\delta^{\mathrm{mid}}} \frac{\frac{1}{N}}{\lambda_{\mathrm{mid}}\chi - z} + \sum_{\lambda_i \in \Lambda_\delta^{\mathrm{low}}} \frac{\frac{1}{N}}{\lambda_N\chi - z}$$

where $\chi = 1 - \nu - \nu z\hat{s}(z)$ and $\lambda_{\mathrm{mid}} = (\lambda_N/2)(1+\sqrt{\nu}+(1+\sqrt{\nu})^2)$. Thus, all the eigenvalues in $\Lambda_\delta^{\mathrm{mid}}$ and $\Lambda_\delta^{\mathrm{low}}$ are replaced by λ_{mid} and λ_N respectively.

Forming the polynomial in $\hat{s}$ and solving for its roots (solved numerically, as the degree of the polynomial is almost always greater than 4), the SCM density can be obtained as

$$\widehat{p}(\lambda) = \lim_{\epsilon \to 0^+} \frac{1}{\pi}\mathrm{Im}[\hat{s}^*(\lambda + \imath\epsilon)] \tag{15}$$

where $\hat{s}^*(z)$ is the appropriate root of the polynomial that has a non-negative imaginary component (since the density is always non-negative).

Fig. 2 shows the approximate eigenvalue density for the SCM when $N = 30$ and $M = 120$ (i.e., $\nu = 1/4$) for different δ and the representative eigenvalues used in the Stieltjes transforms. The distributions are varied in their shapes, depending on the attenuation and approaches the Marčenko-Pastur density for large attenuation ($\delta = 1$). This variation in shape is less pronounced when the number of observations decreases (i.e., $\nu \rightarrow 1$ for the $M \geq N$ scenario) because the spreading of the eigenvalues is much higher, reducing the clustering phenomenon.

V. Conclusions

The asymptotic eigenvalues of noise covariance matrices in diffuse noise fields with attenuation were derived for 2-D and 3-D media. The presence of attenuation decreases the value of the large eigenvalues when compared to the attenuation free case, and also raises the smaller eigenvalues due to the broadening of the bandwidth of the spatial coherence function. The shape of the eigenvalue density of the finite SCM varies with attenuation and this potentially could be used to retrieve medium attenuation properties.

Appendix

The Bessel sequence, $\mathrm{J}_0(2\pi\beta n),\ n \in \mathbb{Z}$ is not absolutely summable and hence the convergence of the error term is not absolute, as in Sections III-A and III-B. Convergence can be demonstrated by computing the error vector $\mathbf{\Sigma u}_i - \lambda_i \mathbf{u}_i$, where λ_i and $\mathbf{u}_i$ are the ith eigenvalue and eigenvector, as in [8, Section II.A]. Following the derivation until [8, Eq. (15)] and taking the ℓ^q-norm instead of the ℓ^2-norm, we get

$$\begin{aligned} \|\mathbf{\Sigma u}_i - \lambda_i \mathbf{u}_i\|_q^q &\leq \frac{2}{N^{q/2}} \sum_{j=1}^{N/2} \sum_{n=j}^{N-j} |\mathrm{J}_0(2\pi\beta n)|^q \\ &\leq \frac{2}{N^{q/2}} \sum_{j=1}^{N/2} \sum_{n=-\infty}^{\infty} |\mathrm{J}_0(2\pi\beta n)|^q \\ &= N^{1-q/2} \sum_{n=-\infty}^{\infty} |\mathrm{J}_0(2\pi\beta n)|^q \qquad (16) \end{aligned}$$

$\mathrm{J}_0(2\pi\beta n)$ can be shown to have a finite ℓ^q-norm for $q > 2$ using the Hausdorff-Young inequality [22]:

Theorem A.1 (Hausdorff–Young Inequality): If f is a function defined on a locally compact Abelian group G with measure μ, and $\mathscr{F}[f]$ its Fourier transform defined on the Pontryagin dual group $\hat{G}$ with measure $\hat{\mu}$, then for $1 \leq p \leq 2$,

$$\|\mathscr{F}[f]\|_q^{(\hat{G},\hat{\mu})} \leq \|f\|_p^{(G,\mu)} \qquad (17)$$

such that $1/p + 1/q = 1$, where $\|\cdot\|_p^{(G,\mu)}$ denotes the norm on the Lebesgue space $\mathcal{L}^p(G,\mu)$.

Here, the function f is $\phi_0^{\mathrm{2D}}(\kappa)$ defined on the unit circle (G, mapped one-to-one to $\kappa \in [-1/2, 1/2)$) with the Lebesgue measure (μ), $\mathscr{F}[f]$ is the sequence $\mathrm{J}_0(2\pi\beta n)$ (the Fourier coefficients) defined on the set of integers ($\hat{G}$) with a counting measure ($\hat{\mu}$). Applying Theorem A.1 and using (8) for $\beta \leq 1/2$ gives:

$$\begin{aligned} \left(\sum_{n=-\infty}^{\infty} |\mathrm{J}_0(2\pi\beta n)|^q \right)^{p/q} &\leq \int_{-1/2}^{1/2} \left|\phi_0^{\mathrm{2D}}(\kappa)\right|^p \mathrm{d}\kappa \\ &= \frac{2}{\pi^p} \int_0^{\beta} \frac{\mathrm{d}\kappa}{(\beta^2 - \kappa^2)^{p/2}} \\ &= \pi^{-p} \beta^{1-p} \mathcal{B}\left(\frac{1}{2}, 1 - \frac{p}{2}\right) \quad (18) \end{aligned}$$

where $\mathcal{B}(x,y)$ is the Beta function (using the substitution $\kappa = \beta\sqrt{t}$). The RHS of (18) is bounded for $p < 2$ but not for $p = 2$, since $\mathcal{B}(x,y) \rightarrow \infty$ as $y \rightarrow 0$). Hence, due to Parseval's theorem, $\mathrm{J}_0(2\pi\beta n)$ cannot exist in ℓ^2 (i.e., it is not absolutely square summable) and has a finite ℓ^q-norm only if $q > 2$ (using $1/p + 1/q = 1$).

Thus, the error term in (16) is less than $\mathcal{O}(N^{1-q/2})$ for $q > 2$ and approaches 0 independent of i. This argument is valid for all values of β (accounting for spatial aliasing).

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