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OBJECTIVE FUNCTIONS FOR OCEAN ACOUSTIC INVERSION DERIVED BY LIKELIHOOD METHODS

CHRISTOPH F. MECKLENBRÄUKER
Forschungszentrum Telekommunikation Wien, A-1040 Vienna, Austria
E-mail: cfm@ieee.org

PETER GERSTOFT
Marine Physical Laboratory, Scripps Inst. Oceanography, San Diego CA 92093-0704, USA
E-mail: gerstoft@mpl.ucsd.edu

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Selection of a suitable objective function is an integral part of the inverse problem, and poor selection can have a strong influence on the inverse result. Objective functions are here derived for many practical occasions such as for single frequency and broadband, with and without knowledge of source strength, and with and without the received signal phase. These objective functions are all derived from a unified approach based on maximum likelihood and additive Gaussian noise models. The assumptions for the objective function are thus clear and the resulting estimator has good properties. From a Bayesian point of view, the solution to the inverse problem is the *a posteriori* probability distribution of the unknown parameters, which can be found from the likelihood function. Thus using objective functions based on likelihood functions facilitates computing the *a posteriori* distributions.

1. Introduction

Matched Field Processing or Matched Field Inversion has received much attention in the ocean acoustics literature,[1–3] and references herein. However there does not seem to be much guidance in selecting objective functions. In this paper a class of objective functions derived from an assumption of additive Gaussian errors by use of maximum likelihood principles is presented. Under these assumptions the likelihood functions $\mathcal{L}$ and objective functions ϕ are related:

$$\mathcal{L}(\mathbf{m}) = k \exp\left(-\frac{\phi(\mathbf{m})}{T}\right) \tag{1.1}$$

where T is a scaling parameter[4–6] and k is a normalization factor. The form in Eq. (1.1) is also related to the scaling of the objective functions in simulated annealing and genetic algorithms. $\mathbf{m}$ is the unknown parameters, these are any parameters used by an ocean-acoustic forward model, as geoacoustic, source and receiver parameters.[7] Good values of these parameters are found through optimization using exhaustive search, simulated annealing or genetic algorithms. The forward models, the objective functions and the search methods are conveniently combined in one program package SAGA.[8]

The assumptions leading to these objective functions are thus clear. As they are derived from maximum likelihood principles the resulting estimators have good properties. They are consistent, asymptotically Gaussian distributed, and asymptotically efficient. Thus, they converge to the true value for a large number of data samples: The bias disappears asymptotically and the variance of the estimator approaches zero. Moreover, no other bias-free estimator exists with a smaller variance in the limit for a large number of data samples. The Cramer–Rao Bound is asymptotically tight. This means that no other ambiguity surface has stronger curvature at its peak if the peak is located at the true parameter. In that sense, it is the *peakiest* surface. However, the Cramer–Rao Bound is only a local measure, it provides no information about the mainlobe/sidelobe difference of the ambiguity surface.

For synthetic model inversion without errors, e.g., Ref. 9, there is no need for time averaging for getting more stable estimates of the pressure vector, but for real data more stable results are obtained by time-averaging leading either to an average pressure vector or a covariance matrix. Objective functions based on either pressure vector or covariance matrices are considered. Objective functions are derived for many practical occasions such as for single frequency and broadband, with and without knowledge of source strength, and with and without the received signal phase. These objective functions and their assumptions are summarized in Table 1.

The objective functions presented here are not new. In fact, they are all available in SAGA. Discussion of likelihood and objective functions from a signal processing point of view was presented in Refs. 10 and 11 while a more ocean acoustic point of view was taken in Ref. 2.

Further, the derived likelihood functions can be used in a Bayesian approach for computing the *a posteriori* probability distribution of the unknown parameters which fully characterize the solution to the inverse problem. According to the Bayesian view, the full solution to the inverse problem is the *a posteriori* probability distribution. This is a product of the *a priori* probability distribution for the unknown parameters before the experiment and the likelihood function for the unknown parameters based on the observed data after the experiment. The *a posteriori* probability density σ for the environmental model $\mathbf{m}$ is

$$\sigma(\mathbf{m}) = \mathcal{L}(\mathbf{m})\rho(\mathbf{m}) \tag{1.2}$$

where ρ is the *a priori* probability density of $\mathbf{m}$ (before observing the acoustic pressure) and $\mathcal{L}$ is the likelihood function, i.e., the probability density of the acoustic pressure given $\mathbf{m}$.

Estimation of uncertainties from global methods have been discussed in Refs. 12–16. Evaluation of the *a posteriori* probability distributions requires knowledge of the likelihood function of the data. The practical approach for estimating these *a posteriori* distributions is not trivial as evaluation of these requires evaluation of M-dimensional integrals, where M is the number of unknown parameters. It has been suggested to use importance sampling to evaluate the *a posteriori* probabilities.[15] Importance sampling concentrates the sampling

Table 1. Log-likelihood objective functions.

	Known Source		Error	Observed Data			Objective Function	
Sec.	Mag.	Phase	Distrib.	Mag.	Phase	Distrib.	Single Frequency	Broadband
	yes	yes	$\mathcal{N}^{\mathbb{C}}(\mathbf{0},\, \nu_l \mathbf{I})$	yes	yes	$\mathcal{N}^{\mathbb{C}}(\mathbf{w}_l(\mathbf{m})S_l,\, \nu_l \mathbf{I})$	$\Vert\mathbf{q} - \mathbf{w}(\mathbf{m})S\Vert^2$	
3.1.2	yes	no	$\mathcal{N}^{\mathbb{C}}(\mathbf{0},\, \nu_l \mathbf{I})$	yes	yes	$\mathcal{N}^{\mathbb{C}}(\mathbf{w}_l(\mathbf{m})S_l,\, \nu_l \mathbf{I})$	$\Vert\mathbf{q}\Vert^2 + \vert S\vert^2 \Vert\mathbf{w}(\mathbf{m})\Vert^2 - 2\vert S\vert\, \vert\mathbf{q}^\dagger \mathbf{w}(\mathbf{m})\vert$	Eq. (3.16) or (3.18)
3.1.1	no	no	$\mathcal{N}^{\mathbb{C}}(\mathbf{0},\, \nu_l \mathbf{I})$	yes	yes	$\mathcal{N}^{\mathbb{C}}(\mathbf{w}_l(\mathbf{m})S_l,\, \nu_l \mathbf{I})$	$\Vert\mathbf{q}\Vert^2 - \frac{\vert\mathbf{q}^\dagger \mathbf{w}(\mathbf{m})\vert}{\Vert\mathbf{w}(\mathbf{m})\Vert^2}$, (Bartlett Power)	Eq. (3.11) or (3.7)
3.2.1	yes	n/a	$\mathcal{N}^{\mathbb{C}}(\mathbf{0},\, \nu_l \mathbf{I})$ $\mathcal{N}(\mathbf{0},\, \nu_l \mathbf{I})$	yes	no	$\mathcal{R}(\vert\mathbf{w}_l S_l\vert,\, \nu_l \mathbf{I})$ $\mathcal{N}(\vert\mathbf{w}_l(\mathbf{m})S_l\vert,\, \nu_l \mathbf{I})$	$\Vert\mathbf{q}\Vert^2 + \vert S\vert^2 \Vert\mathbf{w}(\mathbf{m})\Vert^2 - 2\vert S\vert\, \vert\mathbf{q}\vert^\dagger\, \vert\mathbf{w}(\mathbf{m})\vert$	Eq. (3.26)
3.2.2	no	n/a	$\mathcal{N}^{\mathbb{C}}(\mathbf{0},\, \nu_l \mathbf{I})$ $\mathcal{N}(\mathbf{0},\, \nu_l \mathbf{I})$	yes	no	$\mathcal{R}(\vert\mathbf{w}_l S_l\vert,\, \nu_l \mathbf{I})$ $\mathcal{N}(\vert\mathbf{w}_l(\mathbf{m})S_l\vert,\, \nu_l \mathbf{I})$	$\Vert\mathbf{q}\Vert^2 - \left(\frac{\vert\mathbf{q}\vert^\dagger \vert\mathbf{w}(\mathbf{m})\vert}{\Vert\mathbf{w}(\mathbf{m})\Vert}\right)^2$	Eq. (3.29)

Abbreviations & Notation:

n/a	not applicable
ν_l	error power spectral density at frequency ω_l
$\mathbf{q}_l$	observed data: complex acoustic pressure at frequency ω_l
$\mathbf{w}_l$	replica vector: complex acoustic pressure transfer function at frequency ω_l
S_l	complex source signal at frequency ω_l
$\Vert\mathbf{q}\Vert$	the 2-norm of vector $\mathbf{q}$, i.e., $\sqrt{\mathbf{q}^\dagger \mathbf{q}}$
$\vert\mathbf{q}\vert$	the vector composed of the magnitudes of the elements of $\mathbf{q}$
$\mathbf{q}^\dagger$	Hermitian transpose of vector $\mathbf{q}$
$\mathbf{q}^\dagger \mathbf{w}$	scalar product of vectors $\mathbf{q}$ and $\mathbf{w}$
$\mathcal{N}^{\mathbb{C}}(\mathbf{w}S,\, \nu \mathbf{I})$	multivariate complex normal distribution with mean vector $\mathbf{w}S$ and covariance matrix $\nu \mathbf{I}$
$\mathcal{N}(\vert\mathbf{w}S\vert,\, \nu \mathbf{I})$	multivariate real normal distribution with mean vector $\vert\mathbf{w}S\vert$ and covariance matrix $\nu \mathbf{I}$
$\mathcal{R}(\vert\mathbf{w}S\vert,\, \nu \mathbf{I})$	multivariate real Rician distribution, i.e., distribution of the vector of component wise magnitudes of a $\mathcal{N}^{\mathbb{C}}(\mathbf{w}S,\, \nu \mathbf{I})$-distributed complex-valued random vector

in areas where the integrand has the largest contribution to the integrals. In Ref. 15, it was implemented using genetic algorithms.

2. Assumptions on Broadband Data

The relation between the complex-valued acoustic pressure data vector $\mathbf{q}(\omega_l)$ on an N-element hydrophone antenna array and the predicted data $\mathbf{p}(\mathbf{m}, \omega_l)$ at an angular frequency ω_l is described by the model

$$\mathbf{q}(\omega_l) = \mathbf{p}(\mathbf{m}, \omega_l) + \mathbf{e}(\omega_l)\,, \tag{2.1}$$

where $\mathbf{e}(\omega)$ is the error term. The predicted data is given by $\mathbf{p}(\mathbf{m}, \omega_l) = \mathbf{w}(\mathbf{m}, \omega_l)S(\omega_l)$, where the complex deterministic source term $S(\omega_l)$ may be unknown. The replica vector $\mathbf{w}(\mathbf{m}, \omega_l)$ describes the acoustic signal propagation. It is obtained using an acoustic propagation code, e.g., SNAP,[17] and an environmental model parameterized by the vector $\mathbf{m} \in \mathcal{M}$ where $\mathcal{M}$ is the set of permissible parameter vectors. In the following, the abbreviation $\mathbf{q}_l = \mathbf{q}(\omega_l)$, etc., is used where ω_l for $(l = 1, \ldots, L)$ are the processed frequencies.

The errors are assumed to be additive. They stem from many sources: errors in describing the environment, errors in the forward model, instrument and measurements errors, and noise in the data. This error term is assumed complex Gaussian distributed, second-order stationary with zero mean and diagonal covariance matrix $\nu_l\mathbf{I}$. For the measured field "close" to the predicted field this assumption is expected to hold well. A Gaussian distribution of the errors is easier justified in frequency than in time domain because the distribution of the discrete Fourier transform approaches a Gaussian distribution for large samples.[18]

Thus, the data $\mathbf{q}_l$ on the receiving array are also complex Gaussian distributed with mean $\mathbf{p}_l(\mathbf{m})$ and the covariance matrix $\nu_l\mathbf{I}$. Note that the spectral power of the error term is frequency dependent and unknown in general. For the derivation of a Maximum Likelihood (ML) estimate, it further is assumed that the data are uncorrelated across frequency and time-snapshots. The source term S_l varies across time snapshots whereas the error power spectral density ν_l is the same for all snapshots. Under the above assumptions the cross spectral density matrix becomes $\mathbf{R}_l = \mathsf{E}[\mathbf{q}_l\mathbf{q}_l^\dagger] = \mathbf{p}_l(\mathbf{m})\mathbf{p}_l^\dagger(\mathbf{m}) + \nu_l\mathbf{I}$, where † refers to the Hermitian transpose.[a]

Errors $\mathbf{e}(\omega_1)$, $\mathbf{e}(\omega_2)$ at differing frequencies $\omega_1 \neq \omega_2$ are assumed uncorrelated. For large observation times it is a good approximation for the error in the data but this might be violated for deterministic modeling errors (parameter discretization and other numerical errors in the environmental model are always present). Hence the data are uncorrelated across frequency and the data covariance is given by

$$\mathbf{C}_l = \mathsf{E}[(\mathbf{q}_l - \mathbf{p}_l)(\mathbf{q}_k - \mathbf{p}_k)^\dagger] = \begin{cases} \nu_l\mathbf{I}, & \text{for } l = k \\ \mathbf{0}, & \text{for } l \neq k \end{cases}. \tag{2.2}$$

The broadband likelihood function $\mathcal{L}$ is the product of the single frequency likelihoods.

[a] other mathematical notation is explained in Table 1

Equivalently, the broadband log-likelihood function L is the sum over the single frequency log-likelihoods.

3. Broadband Likelihood Functions

The assumptions of the previous section lead to the formulation of likelihood functions for broadband data in the frequency domain.

3.1. *Observed data: Magnitude and phase*

If the magnitude and phase of the acoustic pressure can be observed by the array, we can use the complex representation of the measurements. The basic starting point for formulating the broadband likelihood function is the complex Gaussian density.

3.1.1. *Source magnitude and phase unknown*

Assuming a completely unknown source signal is probably most common in matched field algorithms and leads to the Bartlett Power objective function. Here, a broadband analysis is presented for deriving the log-likelihood function. Starting from Eq. (2.1) and using the Gaussianity of $\mathbf{e}(\omega_l)$ as assumed in Sec. 2, the probability density for a single snapshot in time is given by

$$\mathcal{L}_1(\mathbf{m}, S, \nu) = \prod_{l=1}^{L} (\pi\nu_l)^{-N} \exp\left[-\frac{\|\mathbf{q}_l - \mathbf{w}_l(\mathbf{m})S_l\|^2}{\nu_l}\right]. \tag{3.1}$$

Measurement data $\mathbf{q}_{l,k}$ from multiple snapshots in time $k = 1, \ldots, K$ is incorporated by multiplying the corresponding probability densities (3.1) for each single time-frame. This gives

$$\mathcal{L}_1 = \prod_{k=1}^{K}\prod_{l=1}^{L} (\pi\nu_l)^{-N} \exp\left[-\frac{\|\mathbf{q}_{l,k} - \mathbf{w}_l(\mathbf{m})S_{l,k}\|^2}{\nu_l}\right]. \tag{3.2}$$

The ML estimate $\hat{\mathbf{m}}^{\mathrm{ML}}$ for $\mathbf{m}$ is obtained by jointly maximizing over the signal and error parameters ($S_{l,k}$, $\nu_l \,\forall\, l$, k) and the model parameter vector $\mathbf{m}$. The maximization with respect to $S_{l,k}$ can be obtained in closed form: $\hat{S}_{l,k} = \mathbf{w}_l^\dagger(\mathbf{m})\mathbf{q}_{l,k}/\|\mathbf{w}_l(\mathbf{m})\|^2$. It is seen that $\hat{S}_{l,k}$ depends on $\mathbf{m}$ but not on ν. Inserting this into (3.2) yields

$$\mathcal{L}_2(\mathbf{m}, \nu) = \prod_{l=1}^{L} (\pi\nu_l)^{-NK} \exp\left[-\frac{\phi_l(\mathbf{m})}{\nu_l/K}\right] \tag{3.3}$$

where[b]

$$\phi_l(\mathbf{m}) = \operatorname{tr}\hat{\mathbf{R}}_l - \frac{\mathbf{w}_l^\dagger(\mathbf{m})\hat{\mathbf{R}}_l\mathbf{w}_l(\mathbf{m})}{\mathbf{w}_l^\dagger(\mathbf{m})\mathbf{w}_l(\mathbf{m})} = \frac{1}{K}\sum_{k=1}^{K}\left\|\mathbf{q}_{l,k} - \frac{(\mathbf{w}_l^\dagger(\mathbf{m})\mathbf{q}_{l,k})}{\|\mathbf{w}_l(\mathbf{m})\|^2}\mathbf{w}_l(\mathbf{m})\right\|^2 \tag{3.4}$$

[b] "tr" denotes the trace operation

is the Bartlett objective function and

$$\hat{\mathbf{R}}_l = \frac{1}{K}\sum_{k=1}^{K} \mathbf{q}_{l,k}\mathbf{q}_{l,k}^{\dagger} \tag{3.5}$$

is the usual estimate of the cross spectral density matrix. In the special case of *known* error power spectrum ν, the known constants ν_l can be inserted into (3.3) and $\mathcal{L}_2$ is regarded as likelihood function for the observed data. Often, the additional assumption of *temporally* white errors is used, $\nu_l = \nu_0$

$$\mathcal{L}_2(\mathbf{m},\, \nu_0) = (\pi\nu_0)^{-NKL}\prod_{l=1}^{L} \exp\left[-\frac{\phi_l(\mathbf{m})}{\nu_0/K}\right]. \tag{3.6}$$

The corresponding log-likelihood function *sums the Bartlett powers* over all frequencies.

$$\Lambda_2(\mathbf{m},\, \nu_0) = -\frac{K}{\nu_0}\sum_{l=1}^{L} \phi_l(\mathbf{m}) - \underbrace{NKL\,\log(\pi\nu_0)}_{\text{const}} \tag{3.7}$$

If, however, the error spectral density is *unknown* then ν_l is treated as a nuissance parameter and $\mathcal{L}_2$ in (3.3) must be optimized with respect to ν_l. Solving for $\partial\mathcal{L}_2/\partial\nu_l = 0$, the ML error estimate is obtained

$$\hat{\nu_l}^{\mathrm{ML}}(\mathbf{m}) = \frac{1}{N}\phi_l(\mathbf{m})\,. \tag{3.8}$$

This maximizing solution is then inserted into (3.3) giving the contracted form

$$\mathcal{L}_3(\mathbf{m}) = \mathcal{L}_2(\mathbf{m},\, \hat{\nu_l}^{\mathrm{ML}}(\mathbf{m})) = \left(\frac{N^K}{\mathrm{e}\pi^K\bar{\phi}(\mathbf{m})}\right)^{NL} \tag{3.9}$$

where $\bar{\phi}$ is the geometric mean of (3.4) over frequency

$$\bar{\phi}(\mathbf{m}) = \sqrt[L]{\prod_{l=1}^{L} \phi_l(\mathbf{m})}\,. \tag{3.10}$$

It is known that the noise and error spectra are not constant with frequency, see e.g., the frequency variation of the noise spectrum.[19,20]

Thus, Eq. (3.9) is preferred to (3.6). The corresponding log-likelihood function *sums the logarithms of the Bartlett powers* over all frequencies

$$\Lambda_3(\mathbf{m}) = \log\, \mathcal{L}_2(\mathbf{m},\, \hat{\nu_l}^{\mathrm{ML}}(\mathbf{m})) = -NK\sum_{l=1}^{L} \log\, \phi_l(\mathbf{m}) + \text{const} \tag{3.11}$$

Thus, the log-likelihood functions $\Lambda_2(\mathbf{m},\, \nu_0)$ and $\Lambda_3(\mathbf{m})$ differ substantially in their frequency averaging. Whilst $\Lambda_2(\mathbf{m},\, \nu_0)$ calculates the *arithmetic* mean of Bartlett power over frequencies, $\Lambda_3(\mathbf{m})$ is associated with the *geometric* mean of Bartlett power. It was found empirically that frequency averaging over Bartlett power in log-units gives better sidelobe rejection than averaging over Bartlett power itself.[21] In practice there is not a large difference between Eqs. (3.3) and (3.9) if the error power spectrum does not vary too much over the frequency range of interest and the signal-to-noise ratio is high.

The ML solution $\hat{\mathbf{m}}^{\text{ML}}$ is obtained by maximizing $\mathcal{L}_2$ or $\mathcal{L}_3$ (whichever is appropriate) over all $\mathbf{m} \in \mathcal{M}$. Finally, an overall estimate for the error power spectral density is obtained from (3.8) at the environmental ML solution: $\hat{\nu}_l^{\text{ML}}(\hat{\mathbf{m}}^{\text{ML}})$ and can be re-inserted into the likelihood function (3.3). For simplicity, we consider the error spectral density as known and only keep the free argument $\mathbf{m}$ of the objective function ϕ_l. This approach leads to[15]

$$\mathcal{L}(\mathbf{m}) = \mathcal{L}_2(\mathbf{m}, \hat{\nu}_l{}^{\text{ML}}(\hat{\mathbf{m}}^{\text{ML}})) = \prod_{l=1}^{L} \left[\frac{N}{\pi\phi_l(\hat{\mathbf{m}}^{\text{ML}})}\right]^N \exp\left[-N\frac{\phi_l(\mathbf{m})}{\phi_l(\hat{\mathbf{m}}^{\text{ML}})}\right] . \tag{3.12}$$

However, *a posteriori* densities can also be based on (3.9). The expressions (3.9) and (3.12) differ in their error spectral estimates: In (3.9), the error spectrum is estimated for each geoacoustic model vector $\mathbf{m}$. In (3.12) the global ML estimate of the error spectrum (which was found from optimization) is *a posteriori* estimated and is then (as an *a priori*) applied to all model vectors $\mathbf{m}$.[15] The main results of this section are summarized in Table 1.

3.1.2. *Source magnitude known*

It is assumed here that the magnitude of the source $|S_l|$ is known but the phase of the source is unknown. It is assumed that the source magnitude does not vary among snapshots whereas the phase may be fluctuating. This situation closely resembles an ocean acoustic tomographic experiment. It has been found in simulations that this objective gave better results for the electromagnetic refractivity estimation.[22] It is expected that the objective function for this case could often give better accuracy than the Bartlett objective function which is commonly used in ocean acoustic tomography.

After maximizing the likelihood (3.2) over all possible source phases, the ML estimate $\hat{S}_{l,k}^{\text{ML}}$ of the complex source signal at frequency ω_l and snapshot k is obtained

$$\hat{S}_{l,k}^{\text{ML}} = |S_l|\frac{\mathbf{w}_l^\dagger(\mathbf{m})\mathbf{q}_{l,k}}{|\mathbf{w}_l^\dagger(\mathbf{m})\mathbf{q}_{l,k}|} . \tag{3.13}$$

Inserting this solution into the likelihood function results in the following single-frequency log-likelihood objective function[22]

$$\phi_{l,3}(\mathbf{m}) = \frac{1}{K}\sum_{k=1}^{K}(\|\mathbf{q}_{l,k}\|^2 + |S_l|^2\|\mathbf{w}_l(\mathbf{m})\|^2 - 2|S_l|\,|\mathbf{q}_{l,k}^\dagger\mathbf{w}_l(\mathbf{m})|) . \tag{3.14}$$

This can also be expressed as

$$\phi_{l,3}(\mathbf{m}) = \operatorname{tr}\hat{\mathbf{R}}_l - 2|S_l|\frac{1}{K}\sum_{k=1}^{K}|\mathbf{q}_{l,k}^\dagger\mathbf{w}_l(\mathbf{m})| + |S_l|^2\|\mathbf{w}_l(\mathbf{m})\|^2 . \tag{3.15}$$

Just as in the preceding section, different error situations can be distinguished: Firstly, the error spectral density could be known theoretically or from a previous measurement: This leads to weighted arithmetic averaging of $\phi_{l,k,3}(\mathbf{m})$ over frequencies and the corresponding broadband log-likelihood reads

$$\Lambda_{2,3}(\mathbf{m}) = -\sum_{l=1}^{L} \frac{\phi_{l,3}(\mathbf{m})}{\nu_l} - \underbrace{NK \sum_{l=1}^{L} \log(\pi\nu_l)}_{\text{const}} . \tag{3.16}$$

Secondly, the error spectrum can sometimes be assumed flat over the frequency range of interest, i.e., $\nu_l = \nu_0$ for all l. Finally, an arbitrary unknown spectral density $\nu(\omega) > 0$ can be allowed and estimated from the data

$$\hat{\nu}_{l,3}^{\text{ML}} = \frac{1}{N}\phi_{l,3}(\mathbf{m}) . \tag{3.17}$$

giving a broadband log-likelihood function which is very similar to (3.11)

$$\Lambda_{3,3}(\mathbf{m}) = -N \sum_{l=1}^{L} \log\, \phi_{l,3}(\mathbf{m}) + \text{const} \tag{3.18}$$

3.2. *Observed data: magnitude only*

For some types of ocean acoustic experiments, the phase of the acoustic pressure data is not available for inversion. Geoacoustic parameters are often estimated from acoustic pressure magnitude alone, see for example Refs. 22–25. The likelihood function is substantially different from the previous cases. Knowledge of the source phase is irrelevant in this case because it cannot improve the geoacoustic parameter estimates. There are two possible cases: either the source signal magnitude is known or unknown. These cases are treated separately.

3.2.1. *Source magnitude known*

For calculating the likelihood function, the probability density of the acoustic pressure magnitude is needed. For clarity of presentation, the single sensor case is treated first. The probability density of the acoustic pressure amplitude $r = |q_{l,k,n}|$ at frequency ω_l, snapshot k, and sensor n is obtained by expressing $q_{l,k,n}$ (which are $\mathcal{N}^{\mathbb{C}}(\mathbf{w}S,\, \nu\mathbf{I})$-distributed) by amplitude and phase $(r,\, \psi)$. Transformation of variables from the real and imaginary part to the amplitude and phase $(r,\, \psi)$ shows that the joint $(r,\, \psi)$ probability density is given by

$$f_{l,k,n}(r,\, \psi) = \frac{r}{\pi\nu_l} \exp\left[-\frac{\|re^{j\psi} - w_{ln}S_l\|^2}{\nu_l}\right] . \tag{3.19}$$

Here, $w_{ln} = w_{ln}(\mathbf{m})$ denotes the nth component of the replica vector $\mathbf{w}_l = \mathbf{w}_l(\mathbf{m})$. In a second step, the phase is averaged out

$$f_{l,k,n}(r) = \int_{-\pi}^{\pi} f_{l,k,n}(r,\, \psi)\frac{\mathrm{d}\psi}{2\pi}$$
$$= \frac{r}{\pi\nu_l}\exp\left(-\frac{r^2 + |w_{ln}S_l|^2}{\nu_l}\right)\int_{-\pi}^{\pi}\exp\left(\frac{2r|w_{ln}S_l|\cos\,\psi}{\nu_l}\right)\frac{\mathrm{d}\psi}{2\pi}$$

The remaining integral is recognized as the modified Bessel function of first kind and zero order, i.e., $\mathrm{I}_0(2r|w_{ln}S_l|/\nu_l)$. This gives a Rician probability density for the acoustic amplitude.[26] The symbol $\mathcal{R}(|\mathbf{w}_l S_l|,\, \nu_l\mathbf{I})$ is introduced for specifying the multivariate Rician probability density which describes the statistics of the component-wise magnitude vector of a complex Gaussian $\mathcal{N}^{\mathbb{C}}(\mathbf{w}_l S_l,\, \nu_l\mathbf{I})$-distributed random vector. For a single snapshot, the broadband likelihood function for an N-element array is given by a product of Rician densities

$$\mathcal{L}_R(\mathbf{m}|S,\, \nu) = \prod_{l=1}^{L}\mathcal{R}(|\mathbf{w}_l S_l|,\, \nu_l\mathbf{I})$$
$$= \prod_{l=1}^{L}\frac{Q}{\nu_l^N}\exp\left[-\frac{\|\mathbf{q}_l\|^2 + |S_l|^2\|\mathbf{w}_l\|^2}{\nu_l}\right]\prod_{n=1}^{N}\mathrm{I}_0\left(\frac{2|w_{ln}S_l q_{l,1,n}|}{\nu_l}\right) \qquad (3.20)$$

The constant scaling factor Q does not depend on the parameter vector $\mathbf{m}$ and is omitted below. Taking the logarithm of (3.20) yields the broadband log-likelihood function

$$\Lambda_R(\mathbf{m}|S,\, \nu) = \sum_{l=1}^{L}\left[-N\,\log\,\nu_l - \frac{\|\mathbf{q}_l\|^2 + |S_l|^2\|\mathbf{w}_l\|^2}{\nu_l} + \sum_{n=1}^{N}\log\mathrm{I}_0\left(\frac{2|w_{ln}S_l q_{l,1,n}|}{\nu_l}\right)\right] \qquad (3.21)$$

If the error spectrum ν_l is known, Λ_R can be used directly as a broadband objective function. However, if the error spectrum is unknown, it needs to be estimated. Taking the partial derivative of (3.21) with respect to ν_l gives

$$\frac{\partial\Lambda_R}{\partial\nu_l} = \frac{-N\nu_l + \|\mathbf{q}_l\|^2 + |S_l|^2\|\mathbf{w}_l\|^2 - 2|S_l|\sum_{n=1}^{N}|w_{ln}q_{l,1,n}|\Upsilon\left(\frac{2|w_{ln}S_l q_{l,1,n}|}{\nu_l}\right)}{\nu_l^2}. \qquad (3.22)$$

Here the logarithmic derivative is defined

$$\Upsilon(x) = \frac{\mathrm{d}\,\log\,\mathrm{I}_0(x)}{\mathrm{d}x} = \frac{\mathrm{I}_1(x)}{\mathrm{I}_0(x)} \qquad (3.23)$$

which is a smooth and increasing function of x and $0 \le \Upsilon(x) < 1$. By equating (3.22) to zero and solving it numerically, the ML estimate for the error spectral power is found. Additional insight is gained by discussing the limiting case of high signal-to-noise ratio analytically.

For high SNR, Eq. (3.22) can be simplified by setting $\Upsilon(x) \to 1$ which is equivalent to approximating $\mathrm{I}_0(x) \to \exp(x)$. This leads to

$$\hat{\nu}_l^{\mathrm{ML}}(\mathbf{m}) = \frac{1}{N}\phi_{l,1}(\mathbf{m}) \tag{3.24}$$

with the definition

$$\phi_{l,1}(\mathbf{m}) = \|\mathbf{q}_l\|^2 + |S_l|^2\|\mathbf{w}_l(\mathbf{m})\|^2 - 2|S_l|\sum_{n=1}^{N} |q_{l,1,n}|\,|w_{l,n}(\mathbf{m})|\,. \tag{3.25}$$

Now, the ML estimate $\hat{\nu}_l^{\mathrm{ML}}$ can be reinserted into the broadband log-likelihood (3.21) giving the high SNR broadband log-likelihood

$$\Lambda_R(\mathbf{m}|S,\, \hat{\nu}_l^{\mathrm{ML}}(\mathbf{m})) = -\frac{N}{2}\sum_{l=1}^{L} \log\, \phi_{l,1}(\mathbf{m}) - \frac{1}{2}\sum_{l=1}^{L}\sum_{n=1}^{N} \log|w_{l,n}(\mathbf{m})| + \mathrm{const} \tag{3.26}$$

The broadband log-likelihood function (3.26) is a sum of two terms: The first term describes an incoherent frequency summing of log $\phi_{l,1}(\mathbf{m})$ and the second term is a penalty function of $\mathbf{m}$ which does not depend on the observed data. The single-frequency case is included in (3.26) for $L = 1$. In Ref. 22, the $\phi_{l,1}(\mathbf{m})$ is used as an objective function for matched field processing based on wavefield magnitude.

The first and second term in Eq. (3.25) is the total energy of the observed and predicted field at the array, respectively. The energy of the predicted field depends mainly on the attenuation of the field. The last term in (3.25) is a scalar product of the observed and predicted magnitude vectors. It depends on the detailed interference patterns of the data and predicted field.

It is interesting to note that the objective function (3.25) can also be derived as the least squares estimate for the magnitude data, i.e., least squares fitting of $|q|$ to $|w|$. This would be an ML approach if the error resulted in zero-mean Gaussian errors for the magnitude of the field. And it would be unbiased if the errors for the magnitude of the field were symmetrical around zero. These assumptions become true for high SNR.

3.2.2. *Source magnitude unknown*

The high SNR case is treated exclusively. The assumption on the received data are the same as in Sec. 3.2.1, thus the magnitude of the received data is also Rician distributed and the derived likelihood and objective function are the same except that the unknown source magnitude $|S_l|$ needs to be estimated from the data. Taking the derivative of Eq. (3.25) with respect to $|S_l|$ leads to the estimate

$$|\widehat{S}_l| = \frac{|\mathbf{q}_l|^\dagger|\mathbf{w}_l(\mathbf{m})|}{\|\mathbf{w}_l(\mathbf{m})\|^2} \tag{3.27}$$

and the reduced objective function becomes

$$\phi_{l,4}(\mathbf{m}) = \min_{|S_l|}\, \phi_{l,1}(\mathbf{m}) = \|\mathbf{q}_l\|^2 - \left(\frac{|\mathbf{q}_l|^\dagger|\mathbf{w}_l|}{\|\mathbf{w}_l\|}\right)^2 \tag{3.28}$$

Among all objective functions in this paper, this one uses the least amount of information: It uses only the magnitude of the observed acoustic pressure and does not require knowledge about the source. It can be interpreted by saying that this objective function is fitting the shape of the observed magnitude curve to the shape of the replica: It tries to fit relative magnitudes. The high SNR broadband log-likelihood is given by (3.26) when replacing $\phi_{l,1}(\mathbf{m})$ by $\phi_{l,4}(\mathbf{m})$

$$\Lambda_R(\mathbf{m}|S,\, \hat{\nu}_l^{\text{ML}}(\mathbf{m})) = -\frac{N}{2}\sum_{l=1}^{L} \log\, \phi_{l,4}(\mathbf{m}) - \frac{1}{2}\sum_{l=1}^{L}\sum_{n=1}^{N} \log|w_{l,n}(\mathbf{m})| + \text{const}\,. \tag{3.29}$$

4. Conclusion

A consistent choice of objective functions is important for obtaining good inversion results. Clearly the selection of objective functions depends on the properties of both the transmitted signal, the propagation, and the received signal. In the present formulation, the propagation is only indirectly included via the received signal.

A class of objective functions derived from an assumption of additive Gaussian noise by use of maximum likelihood principles is presented. These objective functions are summarized in Table 1 for many practical occasions such as for single frequency and broadband, with and without knowledge of source strength, and with and without the received signal phase. It is expected that usage of these consistently derived likelihood functions gives reasonable inversion results because they incorporate all the available data and *a priori* knowledge about the transmitted signal in a statistically justified way.

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