



KALMAN & PARTICLE FILTERS

AOS SEMINAR

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OUTLINE

- Introduction
- Kalman Framework
- Nonlinear & Non-Gaussian Problem
- Suboptimal Kalman Filters
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 - UKF
- Particle Filter
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- Conclusions



Introduction

$$\mathbf{x}_k = f_{k-1}(\mathbf{x}_{k-1}, \mathbf{v}_k) \quad \text{state equation}$$

$$\mathbf{y}_k = h_k(\mathbf{x}_k, \mathbf{w}_k) \quad \text{measurement equation}$$

\mathbf{x}_k : state vector

\mathbf{y}_k : measurement vector

\mathbf{v}_k : process noise vector

\mathbf{w}_k : measurement noise vector

\mathbf{x} : position and speed of a car, \mathbf{y} : sensor measurement

\mathbf{x} : sea and sediment sound speed profile, \mathbf{y} : acoustic measurement

\mathbf{x} : position of Katrina, \mathbf{y} : seismic measurement

\mathbf{x} : atmospheric refractivity profile, \mathbf{y} : radar clutter measurement

\mathbf{x} : number of whales in the region, \mathbf{y} : visual and acoustic measurements

Bayesian framework

$\mathbf{x}_k, \mathbf{y}_k, \mathbf{v}_k, \mathbf{w}_k$: random variables



Kalman Framework

$$\mathbf{x}_k = f_{k-1}(\mathbf{x}_{k-1}, \mathbf{v}_k) \quad \text{state equation}$$

$$\mathbf{y}_k = h_k(\mathbf{x}_k, \mathbf{w}_k) \quad \text{measurement equation}$$



$$\mathbf{x}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{v}_k \quad \text{state equation}$$

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{w}_k \quad \text{measurement equation}$$

$\mathbf{x}_k, \mathbf{y}_k, \mathbf{v}_k, \mathbf{w}_k$: Gaussian

$\mathbf{F}_k, \mathbf{H}_k$: Linear

Optimal Filter = Kalman Filter

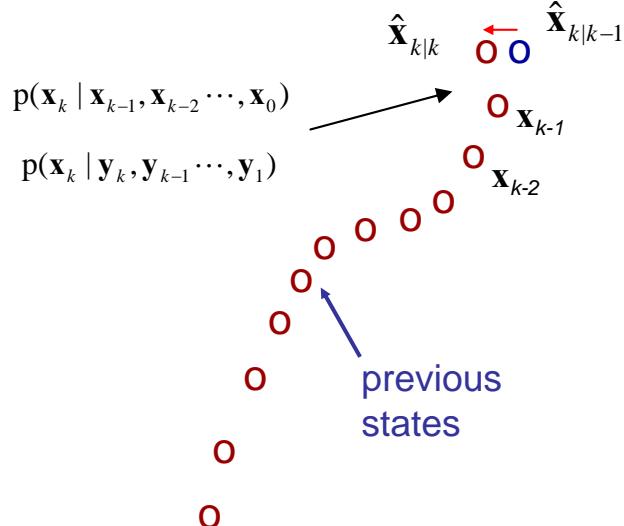
1963



A Single Kalman Iteration

$$\begin{aligned}\mathbf{x}_k &= \mathbf{F}_{k-1} \mathbf{x}_{k-1} + \mathbf{v}_k \\ \mathbf{y}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{w}_k\end{aligned}$$

$$\mathbf{x}_{k|k} \sim \mathcal{N}(\hat{\mathbf{x}}_{k|k}, \mathbf{P}_{k|k})$$



- Predict the mean $\hat{\mathbf{x}}_{k|k-1}$ using previous history.

$$p(\mathbf{x}_k | \mathbf{x}_{k-1})$$

$$\hat{\mathbf{x}}_{k|k-1} = E\{\mathbf{x}_k | \mathbf{x}_{k-1}\} = \int \mathbf{x}_k p(\mathbf{x}_k | \mathbf{x}_{k-1}) d\mathbf{x}_k$$

- Predict the covariance $\mathbf{P}_{k|k-1}$ using previous history.

PREDICT

- Correct/update the mean using new data \mathbf{y}_k

$$p(\mathbf{x}_k | \mathbf{Y}_k)$$

$$\hat{\mathbf{x}}_{k|k} = E\{\mathbf{x}_k | \mathbf{Y}_k\} = \int \mathbf{x}_k p(\mathbf{x}_k | \mathbf{Y}_k) d\mathbf{x}_k$$

- Correct/update the covariance $\mathbf{P}_{k|k}$ using \mathbf{y}_k

UPDATE

$$\cdots \Rightarrow p(\mathbf{x}_{k-1} | \mathbf{Y}_{k-1}) \Rightarrow p(\mathbf{x}_k | \mathbf{Y}_{k-1}) \Rightarrow p(\mathbf{x}_k | \mathbf{Y}_k) \Rightarrow \cdots$$

PREDICTOR-CORRECTOR

DENSITY PROPAGATOR

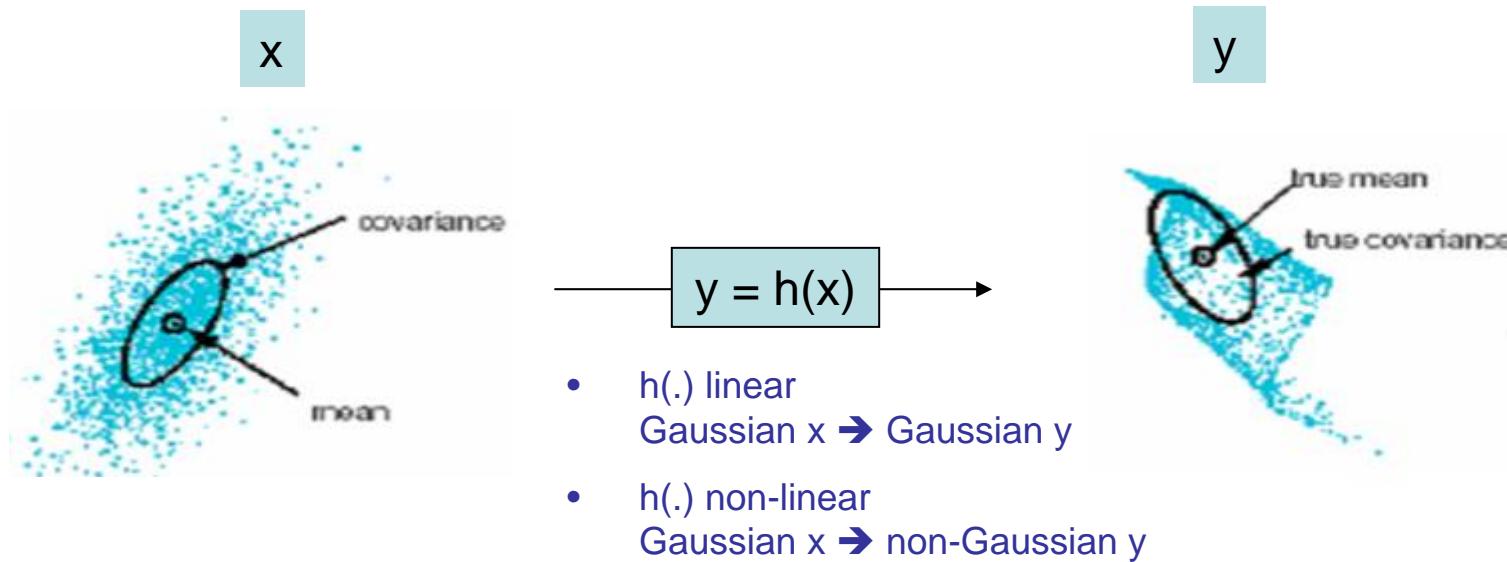
Nonlinear Non-Gaussian

$$\begin{aligned}\mathbf{x}_k &= \mathbf{F}_{k-1} \mathbf{x}_{k-1} + \mathbf{v}_k \\ \mathbf{y}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{w}_k\end{aligned}$$

KF

$$\begin{aligned}\mathbf{x}_k &= f_{k-1}(\mathbf{x}_{k-1}, \mathbf{v}_k) \\ \mathbf{y}_k &= h_k(\mathbf{x}_k, \mathbf{w}_k)\end{aligned}$$

Now What?





Extended Kalman Filter (EKF)



1975

If Nonlinear \rightarrow LINEARIZE!

$$\begin{aligned}\mathbf{x}_k &= f_{k-1}(\mathbf{x}_{k-1}, \mathbf{v}_k) \\ \mathbf{y}_k &= h_k(\mathbf{x}_k, \mathbf{w}_k)\end{aligned}$$

$$\begin{aligned}\mathbf{x}_k &= f_{k-1}(\mathbf{x}_{k-1}) + \mathbf{v}_k \\ \mathbf{y}_k &= h_k(\mathbf{x}_k) + \mathbf{w}_k\end{aligned}$$



$$\begin{aligned}\mathbf{x}_k &= f_{k-1}(\mathbf{x}_{k-1}) + \mathbf{v}_k \cong \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{v}_k \\ \mathbf{y}_k &= h_k(\mathbf{x}_k) + \mathbf{w}_k \cong \mathbf{H}_k\mathbf{x}_k + \mathbf{w}_k\end{aligned}$$

where $\mathbf{H}_k = \left. \frac{\partial h_k}{\partial \mathbf{x}} \right|_{\mathbf{x}_k}$ & $\mathbf{F}_{k-1} = \left. \frac{\partial f_{k-1}}{\partial \mathbf{x}} \right|_{\mathbf{x}_{k-1}}$



$$\begin{aligned}\mathbf{x}_k &= \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{v}_k \\ \mathbf{y}_k &= \mathbf{H}_k\mathbf{x}_k + \mathbf{w}_k\end{aligned}$$

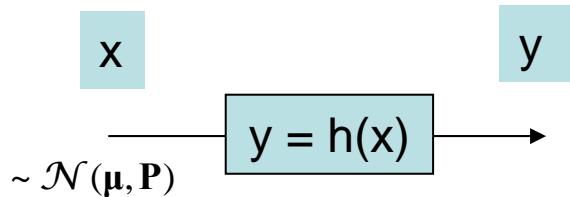
Gaussian and Linear again! \rightarrow now use KF



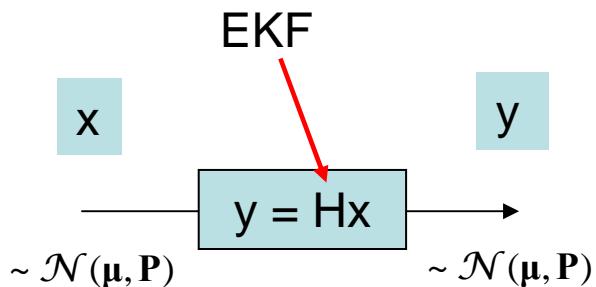
Unscented Kalman Filter (UKF)



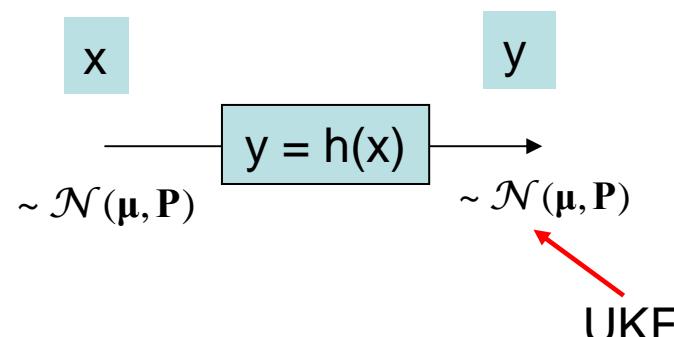
1997



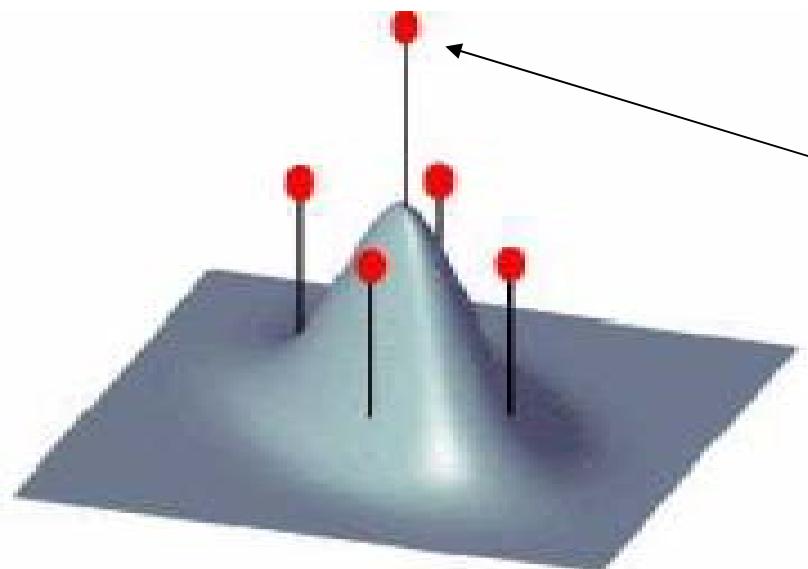
EKF stinks! →



Only first order accurate!



Determine the mean and covariance of \mathbf{y} propagating a minimal number of particles called sigma points through the nonlinear function and reconstruct using the UT.



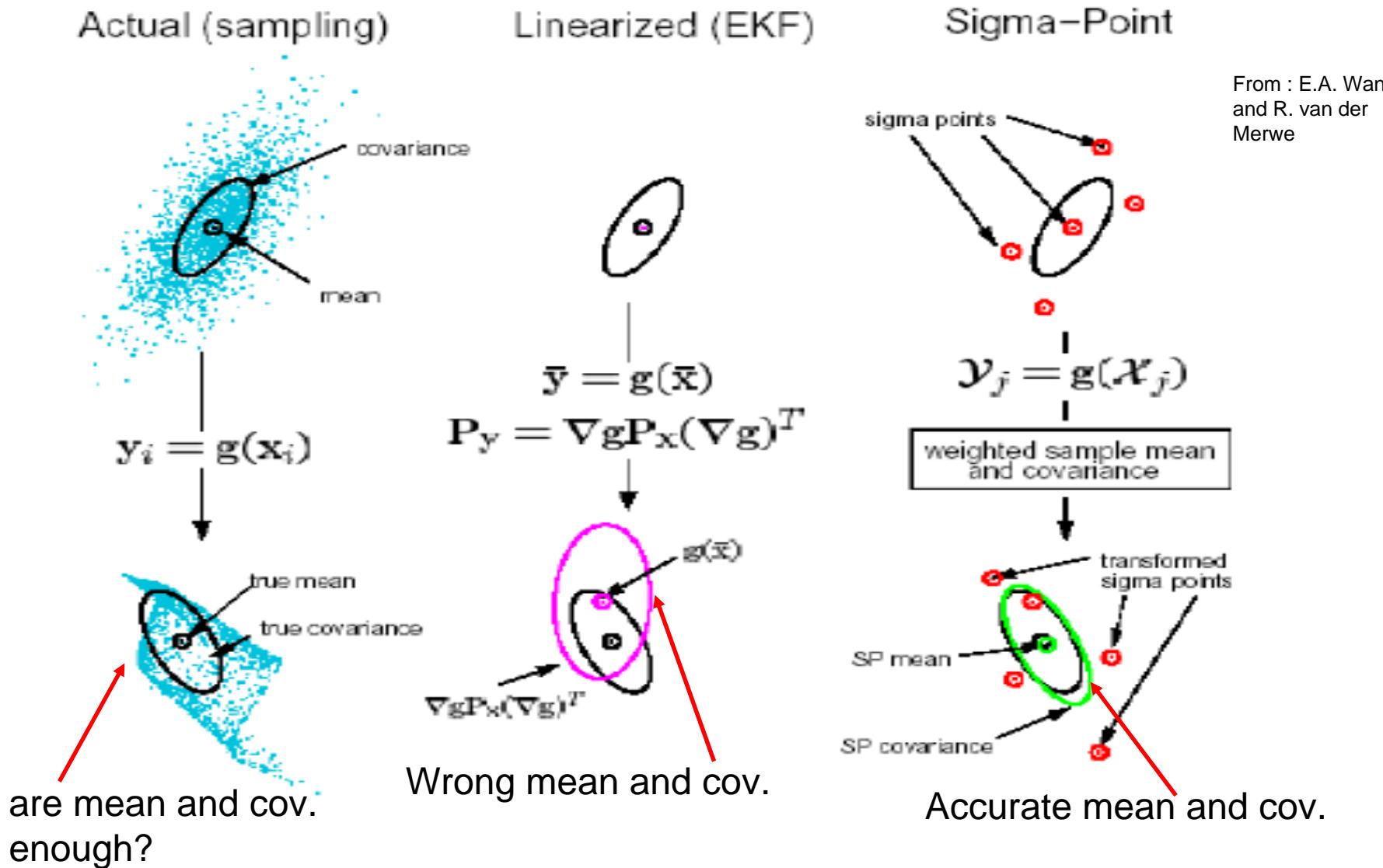
$p(\mathbf{x})$

Sigma points of
the unscented
transform

From : E.A. Wan
and R. van der
Merwe



Unscented Kalman Filter (UKF)





Particle Filter (PF)



POOR MAN'S MONTE CARLO

By J. M. HAMMERSLEY

*Lectureship in the Design and Analysis of Scientific Experiment, University of Oxford;
and Consultant, Atomic Energy Research Establishment, Harwell*

and

1954

K. W. MORTON

Atomic Energy Research Establishment, Harwell

.....but we shall show that Monte Carlo
methods can also serve poor man, who cannot
afford such machinery.....



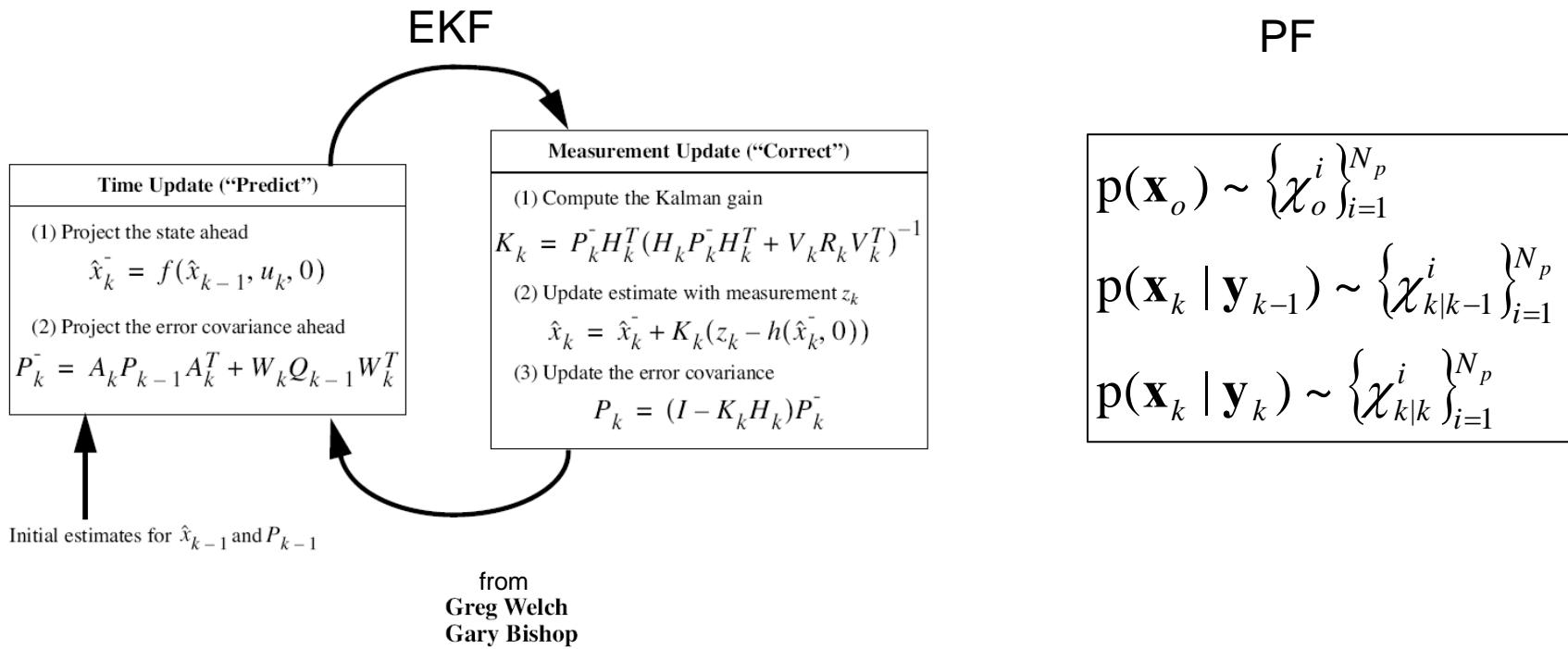
Particle Filter (PF)



$$\begin{aligned}\mathbf{x}_k &= f_{k-1}(\mathbf{x}_{k-1}, \mathbf{v}_k) \\ \mathbf{y}_k &= h_k(\mathbf{x}_k, \mathbf{w}_k)\end{aligned}$$

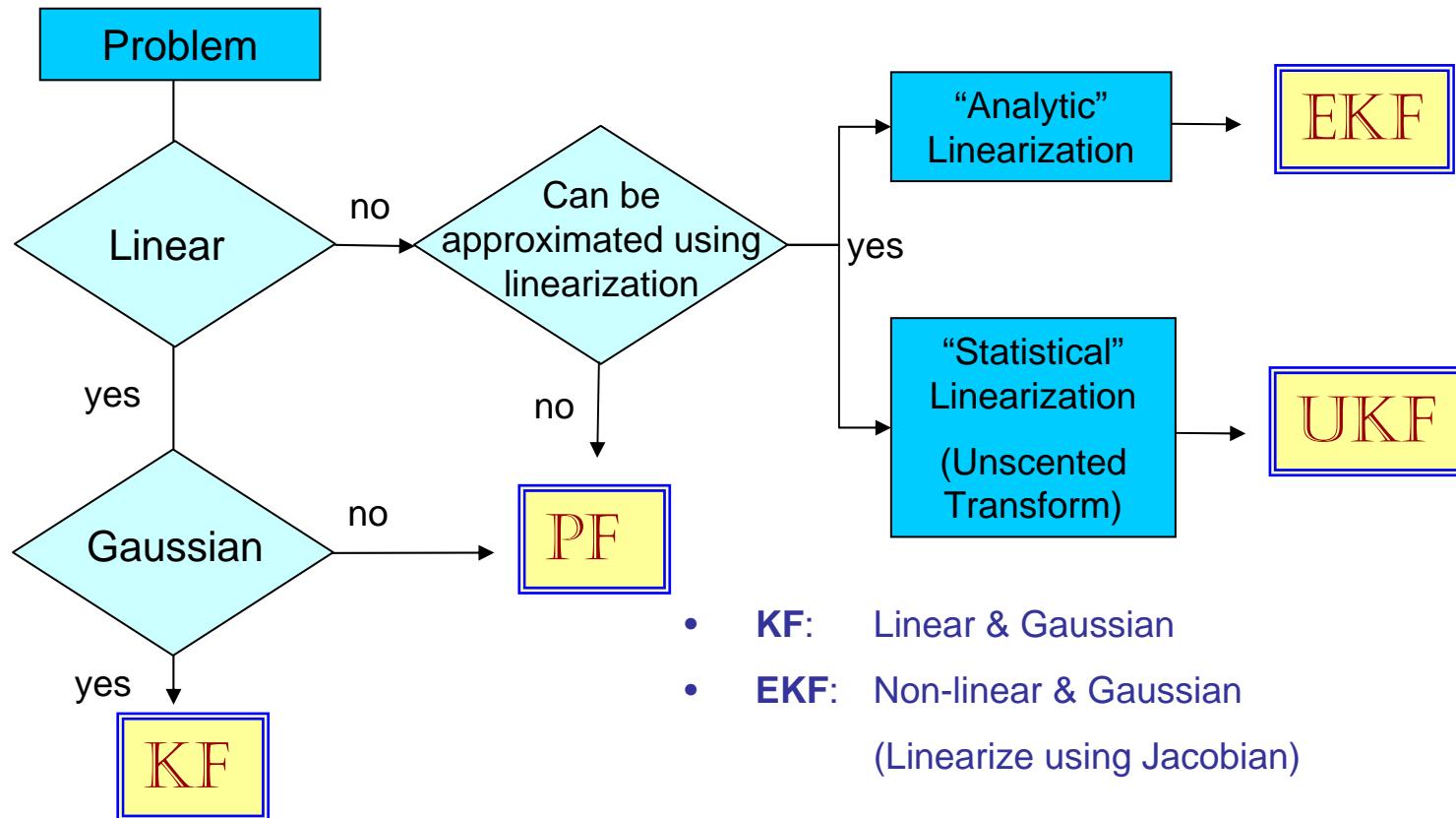
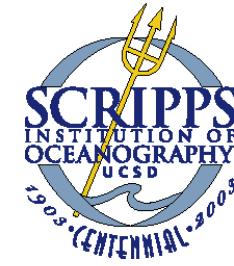
f, h : nonlinear

$\mathbf{x}_k, \mathbf{y}_k, \mathbf{v}_k, \mathbf{w}_k$: non-Gaussian



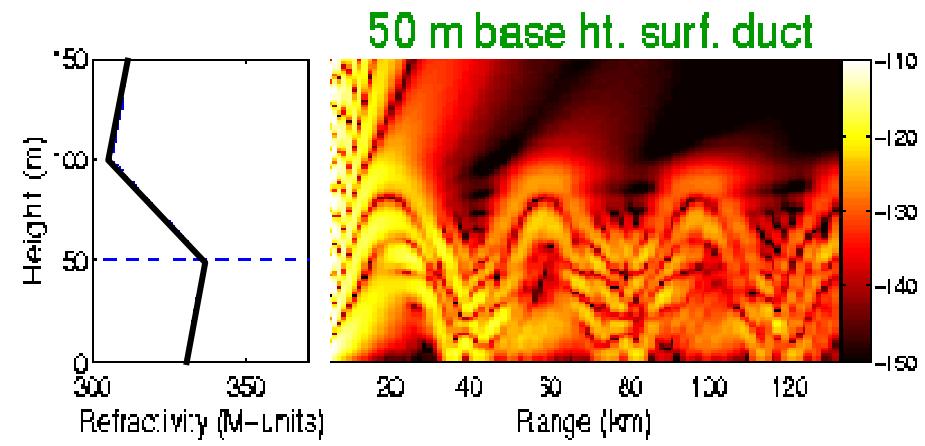
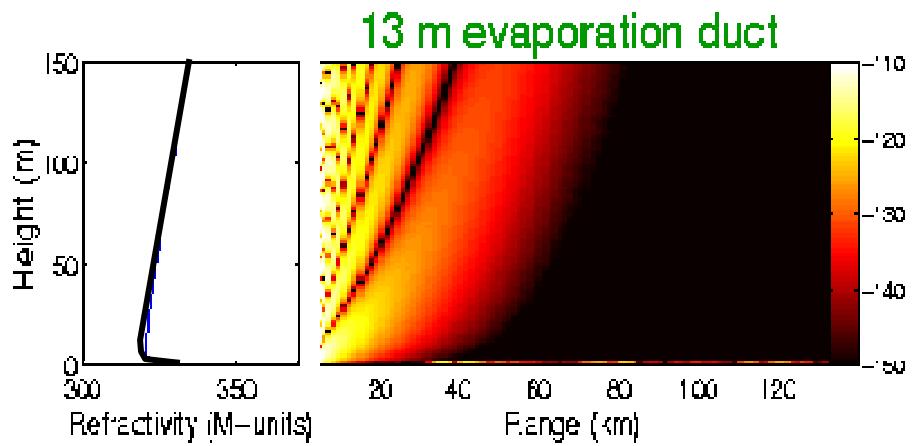


Filter Selection

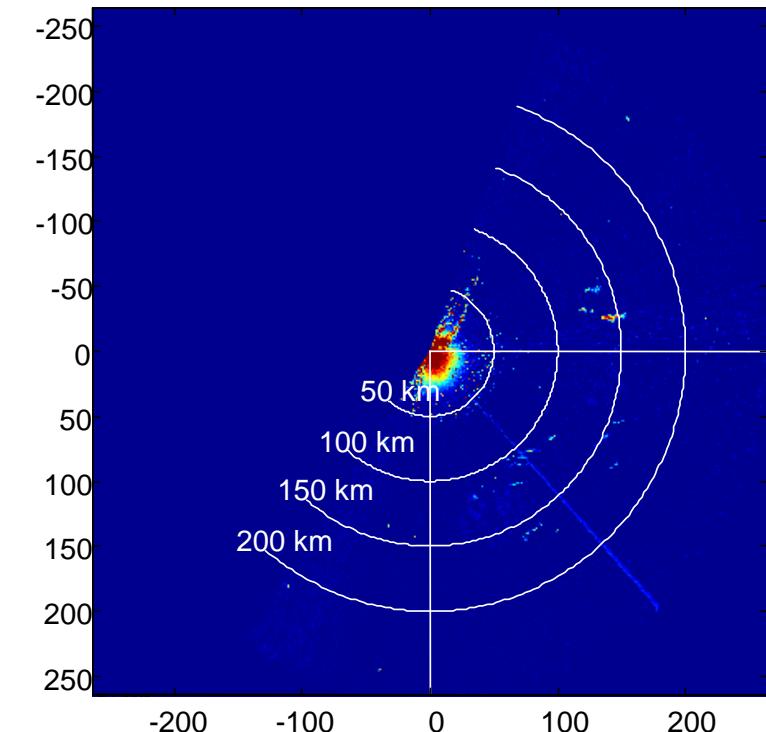


- **KF:** Linear & Gaussian
- **EKF:** Non-linear & Gaussian
(Linearize using Jacobian)
- **UKF:** Non-linear & Gaussian
(Assume Gaussian input – Gaussian output)
- **PF:** Non-linear & Non-Gaussian

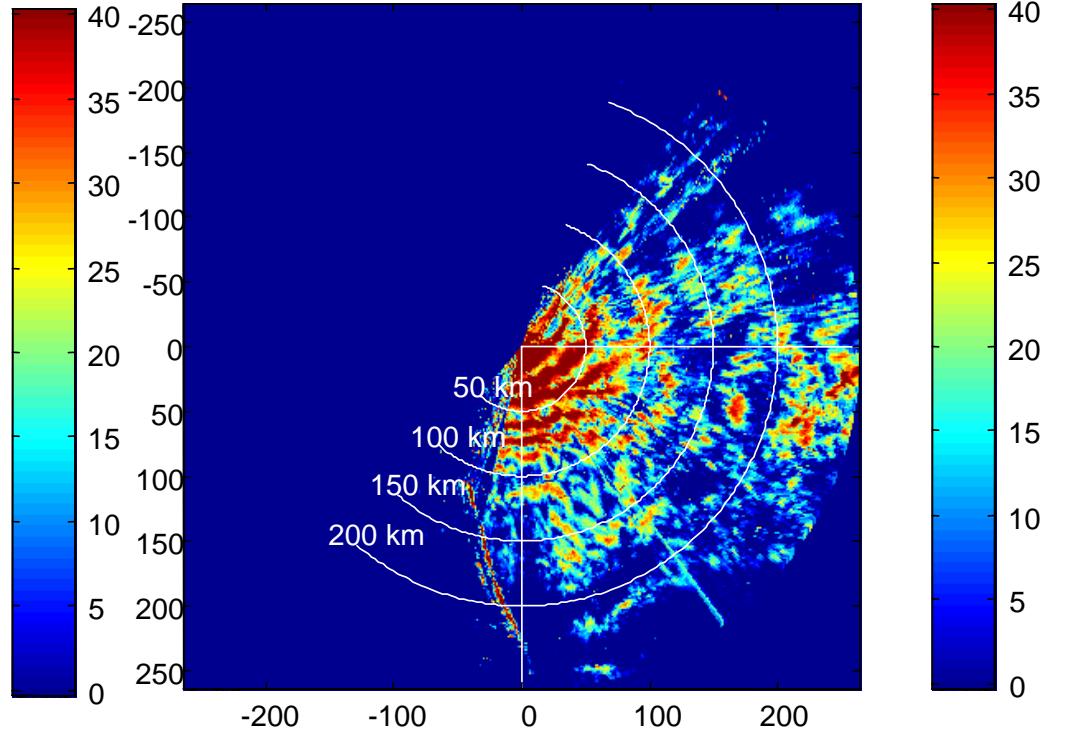
Radar PPI Screen for Evaporative and Surface-Based Ducts



Reflectivity image: March 11, 1998 Map # 031198-20 15:52:33.3

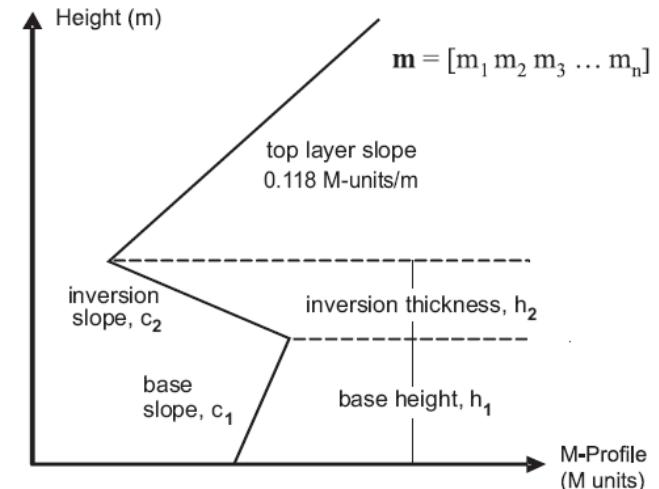
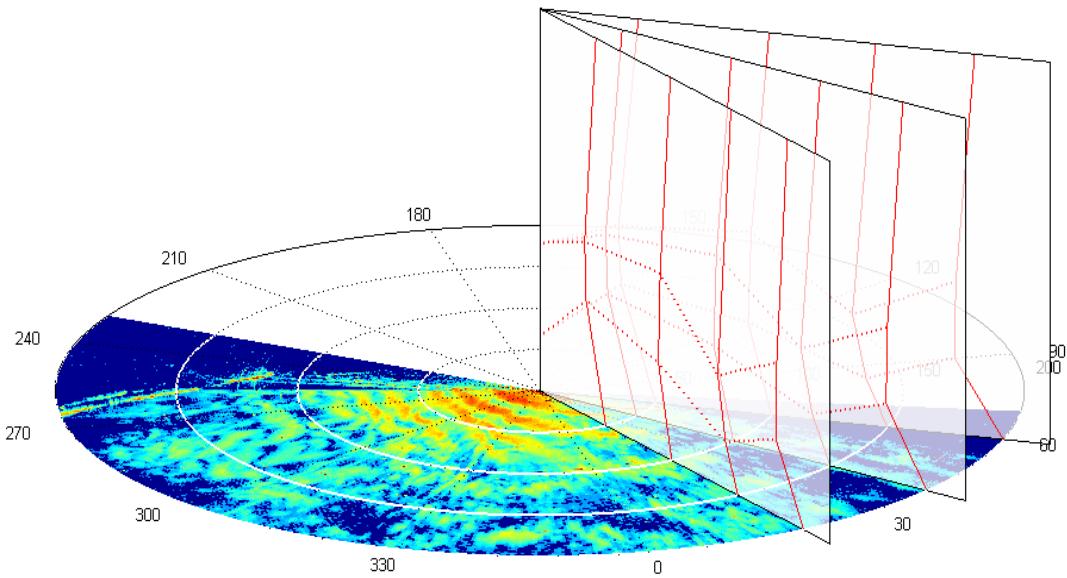


Reflectivity image: April 02, 1998 Map # 040298-17 18:50:00.3





Spatial and Temporal Tracking Using EKF, UKF and PF



M-profile model parameters
(slopes, layer thicknesses)

$$\mathbf{x}_k = \mathbf{x}_{k-1} + \mathbf{v}_k$$

Clutter Power →

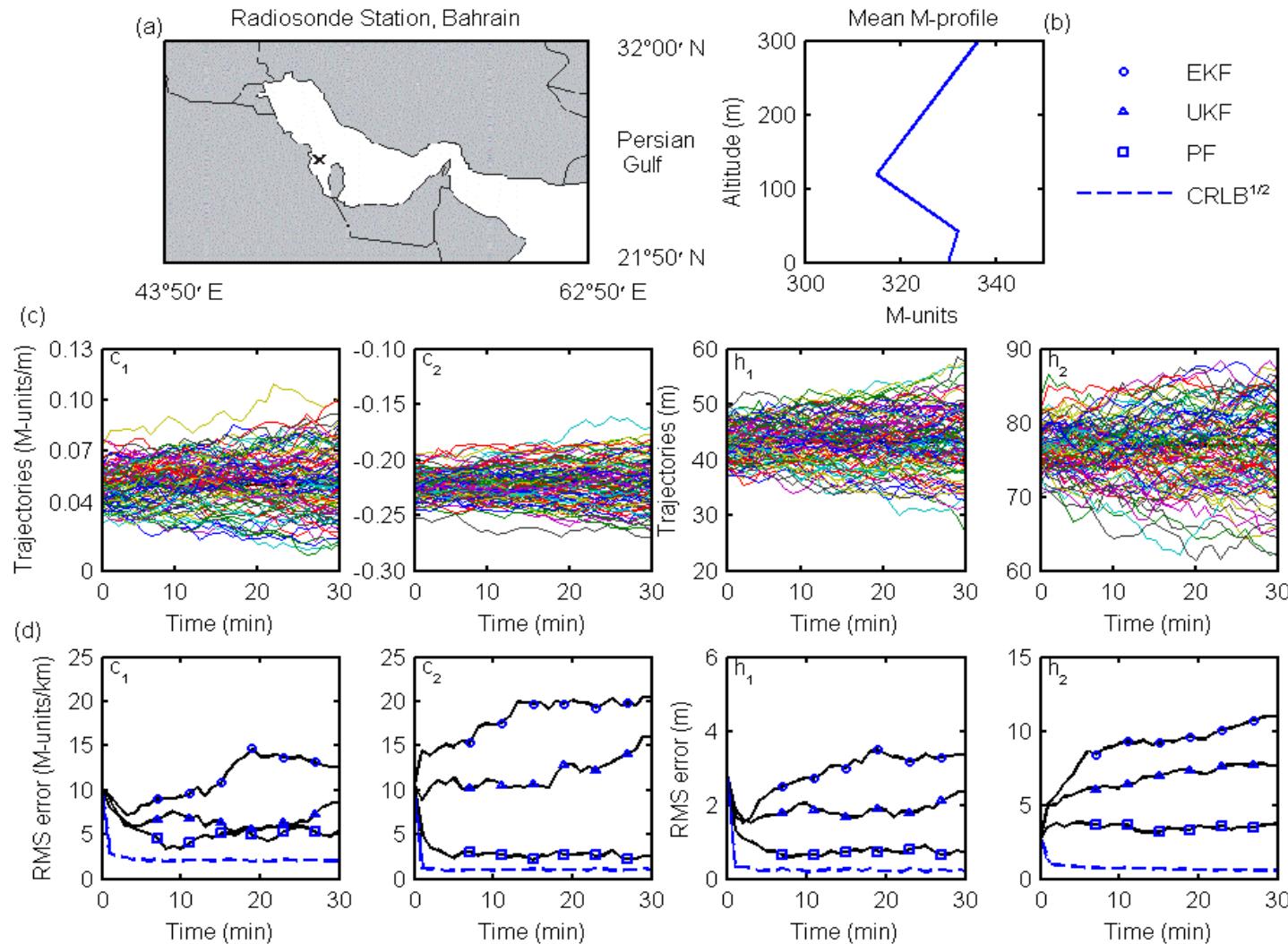
$$\mathbf{y}_k = h(\mathbf{x}_k) + \mathbf{w}_k$$

Parabolic Equation

Sea surface radar
cross section



Tracking Atmospheric Refractivity





Particle Filter (PF)



Method	RMS Error After 30 min.						RTAMS				Average % Improvement Over EKF
	c_1	c_2	h_1	h_2	Avg. Error (%)	Average η (%)	c_1	c_2	h_1	h_2	
	(M-units/km)		(m)				(M-units/km)		(m)		
EKF	12.7	20.5	3.36	11.01	14.2	8	11.7	18.8	3.03	9.48	-
UKF	8.5	16.0	2.33	7.64	9.9	12	6.5	11.9	1.87	6.98	36
PF-200	5.5	2.6	0.71	3.72	4.7	30	4.9	2.7	0.73	3.50	71
PF-1000	3.1	1.9	0.38	0.97	2.3	58	3.6	2.2	0.59	1.96	79
PF-5000	2.0	1.7	0.23	0.96	1.6	77	2.9	2.2	0.46	1.20	84
$\sqrt{\text{CRLB}}$	2.0	1.0	0.21	0.57	1.4	100	2.1	1.0	0.24	0.67	90

Method	Average RTAMS		Divergent Track Percentage After		
	M-units/km	m	10 min.	20 min.	30 min.
EKF	84.2	2.9	47	69	90
UKF	41.8	3.3	0	29	37
PF-200	27.7	0.9	0	0	17
PF-1000	16.2	0.5	0	0	2



Conclusions



- EKF: very easy and fast but not for highly nonlinear and non-Gaussian
- UKF: can handle higher order nonlinearities but still very high nonlinearity and non-Gaussianity is a problem.
- PF: no assumptions. for nonlinear, non-Gaussian problems.

Gaussian sum filters, multiple model filters (interactive MM, static MM, MMPF), Higher-order EKF, IEKF, EnKF, RPF, PF-MCMC, RBPF, SIR, ASIR, HMM filters-Viterbi, and many more.....

THANKS...

Basic Definitions

$$\text{Bayesian MSE} = E\left[\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right)^2\right]$$

MMSE estimator

$$\begin{aligned}\hat{\mathbf{x}}_{k|k}^{MMSE} &\equiv E[\mathbf{x}_k | \mathbf{y}_k, \mathbf{y}_{k-1}, \mathbf{y}_{k-2}, \dots, \mathbf{y}_1] = E[\mathbf{x}_k | \mathbf{Y}_k] \\ &= \int \mathbf{x}_k p(\mathbf{x}_k | \mathbf{Y}_k) d\mathbf{x}_k\end{aligned}$$

$$\hat{\mathbf{x}}_{k|k-1}^{MMSE} \equiv E[\mathbf{x}_k | \mathbf{Y}_{k-1}]$$

MAP estimator

$$\hat{\mathbf{x}}_{k|k}^{MAP} \equiv \arg \max_{\mathbf{x}_k} p(\mathbf{x}_k | \mathbf{Y}_k)$$

$$\boxed{\hat{\mathbf{x}}_{k|k} \equiv E[\mathbf{x}_k | \mathbf{Y}_k]}$$

From now on we will use Bayesian MMSE estimator