

GENERALIZED LIKELIHOOD RATIO TEST FOR SELECTING A GEO-ACOUSTIC ENVIRONMENTAL MODEL

Christoph F. Mecklenbräuer¹

Peter Gerstoft²

Pei-Jung Chung¹

Johann F. Böhme¹

¹ Ruhr University Bochum, D-44780 Bochum/Germany,

² SACLANT Undersea Research Centre, I-19138 La Spezia/Italy
e-mail: cmecklen@rs1.iaee.tuwien.ac.at

ABSTRACT

A generalized likelihood ratio test is considered for testing acoustic environmental models with application to parameter inversion using an acoustic propagation code. In the following, we use the term “hierarchy of models” to denote a sequence of model structures $\mathcal{M}_1, \mathcal{M}_2, \dots$ in which each particular model structure \mathcal{M}_m contains all previous ones as special cases. We propose a combined parameter estimation and multiple sequential test for simultaneously determining the model order and its parameters: *given the observed data, how many parameters should be included in the model?* The last question is important for the order selection problem in hierarchies of models with increasing number of parameters where the observations are corrupted by additive noise. Monte Carlo simulations show the behaviour of the sequential test for selecting a model order as a function of the SNR. Finally, the test is applied to broadband data measured using a vertical array near the island of Elba in the Mediterranean Sea and compared with Akaike’s Information Criterion.

1. INTRODUCTION

This paper deals with statistical hypothesis tests for acoustic environments based on observed data and a replica on a vertical array of sensors. The replica is generated using these environmental model parameters and an acoustic propagation model. The array output is modelled as a superposition of a stationary noise process and the signal of interest. Both the parameter estimation [1] and testing are performed by analysis of data in the frequency domain using a finite Fourier transform.

The definition of objective functions for environmental parameter estimation and choice of test statistics in hypothesis testing using multi-frequency data is still under discussion [2]. The asymptotic Gaussianity of data in frequency domain allows defining approximate log-likelihood functions which are maximized for parameter estimation and used for hypothesis tests based on likelihood quotients [2]. The proposed generalized likelihood ratio test (GLRT) is based on multi-frequency data and exploits the asymptotic Gaussianity of short-time Fourier-transformed measurement data. This test is related to a solution to the detection problem in passive sonar, seismics, and radar applications using a multiple sequential F -test which is based on a frequency-domain regression [3]. The GLRT compares the geometric means over frequency of the estimated noise spectrum under the hypothesis and alternatives. In the case of only one single source, the test statistic is related to the sum of Bartlett powers in dB.

2. PROPAGATION AND DATA MODEL

The SNAP normal-mode code [4] is used to compute the resulting acoustic field from a harmonic point source excitation in a shallow ocean waveguide. The observed acoustic pressure field $X(r, t)$ is sampled at a vertical array of N omni-directional hydrophones over space ($r \in \{r_1, r_2, \dots, r_N\} \subset \mathbb{R}^3$) and time ($t \in \{0, 1, \dots, KT - 1\}$). These samples are treated as a vector process $\underline{X}[t]$, with $X_n[t] = X(r_n, t)$. Let the data model in frequency domain be given by

$$\underline{X}(\omega) = \underline{d}(\omega; \vartheta)S(\omega) + \underline{U}(\omega), \quad (1)$$

where $\underline{d}(\omega; \vartheta)$ is the replica steering vector calculated by SNAP at frequency ω for the geometrical and geo-acoustic parameters summarized in the parameter vector ϑ . Figs. 1 and 2 show the baseline environment with geo-acoustic profiles and relevant physical parameters. This is a distributed-parameter model [5]. The unknown geo-acoustic quantities are distributed in space and must be discretized, before they can be estimated

Let us denote the set of all models by \mathcal{M}^* , i.e. the set of all replica steering vectors \underline{d} which can be implemented by SNAP. The parameter vector ϑ is assumed to be element of an associated parameter set $\mathcal{D}^* \subset \mathbb{R}^r$. The mapping between the parameter set \mathcal{D}^* and the set of models \mathcal{M}^* is termed *model structure*, cf. [5]. Below, the sets \mathcal{M}^* and \mathcal{D}^* will be given a hierarchical build-up.

3. LOG-LIKELIHOOD FUNCTION

The sampled acoustic field $\underline{X}[t] \in \mathbb{R}^N$ is segmented into K snapshots, tapered with a set of L orthonormal windows $w_\ell[t]$ [6, 7], and finite fourier transformed, giving

$$\underline{X}_{k\ell}(\omega) \stackrel{\text{Def}}{=} \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} w_\ell[t] \underline{X}[t+kT] e^{-j\omega t}, \quad \begin{cases} \ell = 0 \dots L-1, \\ k = 0 \dots K-1 \end{cases}$$

The spectral density matrix is estimated by the sample mean,

$$\hat{\mathbf{C}}_{\underline{X}}(\omega) \stackrel{\text{Def}}{=} \frac{1}{KL} \sum_{k=0}^{K-1} \sum_{\ell=0}^{L-1} \underline{X}_{k\ell}(\omega) \underline{X}_{k\ell}^*(\omega). \quad (2)$$

The conditional distribution of $\underline{X}_{k\ell}(\omega)$ given the signal $S_{k\ell}(\omega)$ is known asymptotically: they are approximately independent and identically complex normal distributed, with mean $\underline{d}(\omega; \vartheta)S_{k\ell}(\omega)$, where $S_{k\ell}(\omega)$ is defined similar to $\underline{X}_{k\ell}(\omega)$, and covariance matrix $\mathbf{C}_v = \nu(\omega)\mathbf{I}$, where \mathbf{I} is the identity matrix. Diagonality of the noise term $\nu(\omega)\mathbf{I}$ can be justified by choosing sensor spacing larger than the correlation length of environmental noise. The log-likelihood function for a broadband source is given for this statistical

model by (vectors are underlined, matrices in boldface, * denotes conjugate transpose.)

$$L(\underline{\vartheta}) = -\frac{1}{J} \sum_{j=1}^J \ln(1 - B(\omega_j; \underline{\vartheta})), \quad (3a)$$

$$\text{with } B(\omega; \underline{\vartheta}) = \frac{\underline{d}^*(\omega; \underline{\vartheta}) \hat{\mathbf{C}}_{\mathbf{x}}(\omega) \underline{d}(\omega; \underline{\vartheta})}{|\underline{d}(\omega; \underline{\vartheta})|^2 \text{tr} \hat{\mathbf{C}}_{\mathbf{x}}(\omega)} \quad (3b)$$

A set of frequencies $\{\omega_1, \dots, \omega_J\}$ covering the spectrum of the acoustic source is selected in order to minimize the computational costs associated with the evaluation of (3a). It can be shown that (3b) is asymptotically Beta(p, q)-distributed with parameters $p = KL$ and $q = KL(N-1)$ if $\underline{\vartheta}$ is constant. However, if $\underline{\vartheta}$ is estimated from the observed data ($\underline{\vartheta} \rightarrow \hat{\underline{\vartheta}}$), the parameters of the Beta-distribution are different. Optimization of $L(\underline{\vartheta})$ is implemented by a combined approach of the globally convergent Genetic Algorithm (GA) and locally convergent BFGS updates.

4. ORDERED MODELS AND MULTIPLE SEQUENTIAL GLRT

We define a hierarchy of model sets $\{\mathcal{M}_1, \mathcal{M}_2, \dots\}$ in which each \mathcal{M}_m ($m \in \mathbb{N}$) contains all previous ones as special cases: $\mathcal{M}_m \subset \mathcal{M}_{m+1}$. One can think of the model set \mathcal{M}_m as being obtained from \mathcal{M}_{m+1} by *freezing* some components of $\underline{\vartheta}_{m+1}$ to specific values. The dimension $r_m \in \mathbb{N}$ of the parameter set $\mathcal{D}_m \subset \mathbb{R}^{r_m}$ increases monotonically with m . The model structures are visualized in Fig. 2. The GLRT statistic is defined as the difference in optimized values of the L -functions for each model order. Unfortunately, this direct approach is unfeasible, because the test statistic depends on the parameters. In [8], a workaround with one hypothesis and three alternatives was presented. The test of the *smaller* model \mathcal{M}_m against the *bigger* model \mathcal{M}_{m+1} is implemented by a three-step sequential procedure. For the hierarchy, we will use the following hypotheses $H_{i,m}$ and alternatives $A_{i,m}$, ($i = 1, 2, 3$):

Step 1: $H_{1,m}$: $\underline{X} = \underline{U}$,

$$A_{1,m} : \underline{X} = (\underline{d}_m \ \underline{d}_{m+1}) \begin{pmatrix} S_m \\ S_{m+1} \end{pmatrix} + \underline{U},$$

with $|S_m|^2 + |S_{m+1}|^2 \neq 0$.

Step 2: $H_{2,m}$: $\underline{X} = \underline{d}_m S_m + \underline{U}$,

$$A_{2,m} : \underline{X} = (\underline{d}_m \ \underline{d}_{m+1}) \begin{pmatrix} S_m \\ S_{m+1} \end{pmatrix} + \underline{U},$$

with $|S_{m+1}|^2 \neq 0$ and arbitrary S_m .

Step 3: $H_{3,m}$: $\underline{X} = \underline{d}_{m+1} S_{m+1} + \underline{U}$,

$$A_{3,m} : \underline{X} = (\underline{d}_m \ \underline{d}_{m+1}) \begin{pmatrix} S_m \\ S_{m+1} \end{pmatrix} + \underline{U},$$

with $|S_m|^2 \neq 0$ and arbitrary S_{m+1} .

We have omitted the dependency on ω in notation of all quantities and \underline{d}_m is shorthand for $\underline{d}(\omega, \hat{\underline{\vartheta}}_m)$, with the ML parameter estimate for model \mathcal{M}_m

$$\hat{\underline{\vartheta}}_m = \arg \max_{\underline{\vartheta}_m \in \mathcal{D}_m} L(\underline{\vartheta}_m).$$

The above hypotheses and alternatives are interpreted in words as follows:

$H_{1,m}$	no signal in the data
$H_{2,m}$	model \mathcal{M}_m generated the data
$H_{3,m}$	model \mathcal{M}_{m+1} generated the data
$A_{1,m}$	\mathcal{M}_m or \mathcal{M}_{m+1} generated the data
$A_{2,m}$	some component of the data cannot be adequately modeled by \mathcal{M}_m
$A_{3,m}$	some component of the data cannot be adequately modeled by \mathcal{M}_{m+1}

The proposed simultaneous estimation and test is now formulated in form of a concise algorithm which calculates $\hat{m}, \hat{\underline{\vartheta}}_{\hat{m}}$ from the data X :

```

 $\hat{m} := 1$ 
 $\hat{\underline{\vartheta}}_1 := \arg \max_{\underline{\vartheta}_1 \in \mathcal{D}_1} L(\underline{\vartheta}_1)$ 
for  $m := 1, 2, \dots$  do
   $\hat{\underline{\vartheta}}_{m+1} := \arg \max_{\underline{\vartheta}_{m+1} \in \mathcal{D}_{m+1}} L(\underline{\vartheta}_{m+1})$ 
  if  $H_{1,m}$  cannot be rejected against  $A_{1,m}$  then
     $\hat{m} := m - 1$ , stop "identifiability problem"
  else
    if  $H_{2,m}$  cannot be rejected against  $A_{2,m}$  then
       $\hat{m} := m$ , stop "this is conservative"
    else
      if  $H_{3,m}$  cannot be rejected against  $A_{3,m}$  then
         $\hat{m} := m + 1$ , stop
      else
         $\hat{m} \geq m + 1$ , continue with the loop over  $m$ 
  end of for loop

```

We need the following GLRT statistics for the algorithm ($i = 1, 2, 3$)

$$T_{i,m} = \frac{1}{J} \sum_{j=1}^J \ln \left(1 + \frac{n_1}{n_2} V_{i,m}(\omega_j) \right), \quad \text{with}$$

$$V_{i,m}(\omega) = \frac{n_2 \text{tr}[(\mathbf{P}_{A_m}(\omega_j) - \mathbf{P}_{i,m}(\omega_j)) \hat{\mathbf{C}}_{\mathbf{x}}(\omega)]}{n_1 \text{tr}[(\mathbf{I} - \mathbf{P}_{A_m}(\omega_j)) \hat{\mathbf{C}}_{\mathbf{x}}(\omega)]}. \quad (4)$$

Where we have defined the following projection matrices¹

$$\begin{aligned} \mathbf{P}_{1,m}(\omega) &= \mathbf{0}, \\ \mathbf{P}_{2,m}(\omega) &= \underline{d}_m \underline{d}_m^* / |\underline{d}_m|^2, \\ \mathbf{P}_{3,m}(\omega) &= \underline{d}_{m+1} \underline{d}_{m+1}^* / |\underline{d}_{m+1}|^2, \\ \mathbf{P}_{A_m}(\omega) &= \mathbf{Q}_m \mathbf{Q}_m^* \text{ with: } \mathbf{Q}_m = \text{orth}(\underline{d}_m \ \underline{d}_{m+1}) \end{aligned}$$

The variables $V_{i,m}(\omega)$ which are used in the first step of the test can be seen as estimates for the signal- to noise ratio (SNR) under the hypotheses $H_{1,m}$. The quantities $V_{2,m}(\omega), V_{3,m}(\omega)$ estimate an *increment in SNR* in the data between the models $\underline{d}_m \in \mathcal{M}_m$ and $\underline{d}_{m+1} \in \mathcal{M}_{m+1}$. Thus, they evaluate incremental goodness-of-fit for the two model orders m and $m+1$. The GLRT statistics $T_{i,m}$ are related to the geometric mean of goodness-of-fit over the range of frequencies.

The distributions of the test statistics $T_{i,m}$ do not depend on $\hat{\underline{\vartheta}}_m$ under the hypotheses $H_{i,m}$, since it can be shown that $V_{i,m}(\omega)$ is F_{n_1, n_2} -distributed. The distribution of $T_{i,m}$ is completely known in the single frequency case ($J = 1$), and for the broadband case ($J \geq 3$), the distribution can be

¹orth() denotes orthonormalization of vectors

closely approximated by a Gaussian with mean and variance given by

$$\mu_{i,m} = \Psi\left(\frac{n_1 + n_2}{2}\right) - \Psi\left(\frac{n_2}{2}\right), \quad (5a)$$

$$\sigma_{i,m}^2 = \frac{1}{J} \left(\Psi'\left(\frac{n_2}{2}\right) - \Psi'\left(\frac{n_1 + n_2}{2}\right) \right), \quad (5b)$$

with $\Psi(z) = \frac{d \ln \Gamma(z)}{dz}$ and $\Psi'(z) = \frac{d^2 \Psi(z)}{dz^2}$. The required test thresholds $t_{i,m,\alpha}$ for a false-alarm rate α are approximated by the $(1 - \alpha)$ -quantile of the Gaussian. Higher-order terms of the Edgeworth expansion and more subtle bootstrap methods are not required in this application. This was proven by simulations.

5. DEGREES OF FREEDOM

Up to now, nothing has been said about the role of the degrees of freedom (DOF) of the F_{n_1, n_2} -distribution in the test. This is a delicate problem, due to the existence of both *linear* and *non-linear* unknown parameters in the estimation problem, cf. [2].

$$\begin{aligned} \text{Step 1: } n_1 &= KL(\tau_m + \tau_{m+1} + 4), \\ n_2 &= KL(2N - (\tau_m + \tau_{m+1} + 4)) \end{aligned} \quad (6a)$$

$$\begin{aligned} \text{Steps 2,3: } n_1 &= KL(\tau_m + \tau_{m+1} + 2), \\ n_2 &= KL(2N - (\tau_m + \tau_{m+1} + 4)) \end{aligned} \quad (6b)$$

The DOF can be made independent of τ_m, τ_{m+1} if the two sets of frequencies $\{\omega_j\}$ used for estimating and testing are designed to be disjoint. In this case, the DOF are given by (6ab) for $\tau_m = \tau_{m+1} = 0$.

6. COMPARISON WITH AIC

Akaike's Information Criterion (AIC) represents a more classic approach to the order selection problem in hierarchies of models, cf. [5]. In the present application, the AIC selects the model order $\hat{m} \in \mathbb{N}$ which minimizes the criterion.

$$\text{AIC}[m] = -L(\hat{\vartheta}_m) + \frac{\tau_m}{N}. \quad (7)$$

From this simple form of $\text{AIC}[m]$, we can directly calculate the required increase in likelihood to equalize the cost of additional parameters. The AIC prefers \mathcal{M}_{m+1} over \mathcal{M}_m iff

$$L(\hat{\vartheta}_{m+1}) - L(\hat{\vartheta}_m) > \frac{1}{N}(\tau_{m+1} - \tau_m). \quad (8)$$

7. SIMULATIONS

Figs. 3,4,5 show the results of Monte Carlo simulations. For each SNR, we conduct 50 independent random experiments. The data were generated by a selected model in the largest model structure considered and corrupted by additive noise at prescribed SNR. In each experiment, we calculated the maximum-likelihood estimates and applied the proposed algorithm for the first model structures by global optimization of (3a) using a genetic algorithm, analogous to [1]. Monte-Carlo estimates for the probabilities of the test decisions as a function of SNR are given in Figs. 3 and 4. Typical threshold effects can be clearly observed in the figures: at a characteristic SNR value, the test decides with high probability in favour of the correct model order. In specific intervals of SNR, *coarser/downgraded* models are preferred to the correct solution. Each downgraded lower-order model has its own range in SNR in which it is dominant, indicating identification problems for higher-order models at the corresponding SNR levels. Fig. 5 shows the outcome of the

AIC-criterion. If the improvement in maximum-likelihood between \mathcal{M}_m and \mathcal{M}_{m+1} exceeds $2.083 \cdot 10^{-2}$ per additional parameter, \mathcal{M}_{m+1} is preferred.

8. APPLICATION TO NORTH ELBA DATA

The algorithm is applied to experimental acoustic data which were recorded north of Elba island (Italy) [9, 10]. A record of one minute ($KT = 6 \cdot 10^4$) of time samples was used which were divided into $K = 15$ snapshots. The spectral analysis was performed with $L = 4$ windows and an analysis bandwidth of $W = \frac{2.5}{T}$. We selected $J = 14$ frequencies in the range 150...180 Hz for the test. The two frequency sets for estimation and test were designed to be disjoint, see Sec. 5.. The calculated $T_{2,m}$ GLRT statistics for the models $\mathcal{M}_1, \dots, \mathcal{M}_5$ are shown in Fig. 6. This shows that the GLRT selects $\hat{m} \geq 5$ due to the high SNR in the observed data (≈ 30 dB). Conversely, we conclude that the geo-acoustic parameter set \mathcal{D}_5 results in a significantly better model $\hat{d}_5 \in \mathcal{M}_5$ than all lower-order models in Fig. 2.

9. CONCLUSION

The proposed sequential test can be used for designing source power spectra at sea and proving feasibility of geo-acoustic inversions. The test is applied to the real sonar N-Elba data and compared with Akaike's Information Criterion for selecting model orders. A critical threshold SNR value must be maintained at sea trials for inversion of a given set of parameters inside \mathcal{M}^* for significant results. The proposed multiple sequential estimation-and-test algorithm can be used for designing source power spectra at sea and proving feasibility of geo-acoustic inversions at practical signal levels.

ACKNOWLEDGEMENTS

The authors thank NATO SACLANTCEN (La Spezia/Italy) for providing the measurements and F. Seifert for computational resources.

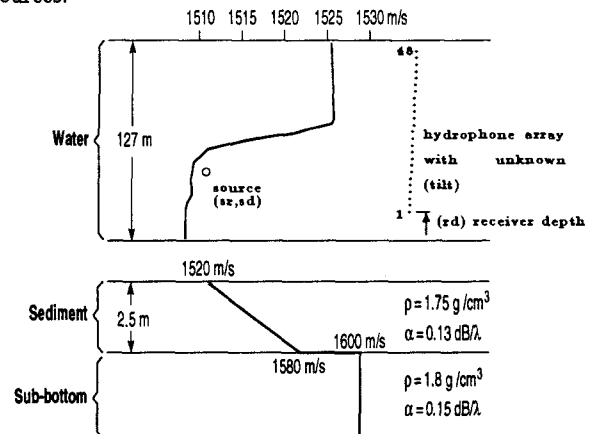


Figure 1. Environmental baseline model of N-Elba site, cf. [10]

REFERENCES

- [1] P. Gerstoft, "Inversion of seismoacoustic data using genetic algorithms and a posteriori distributions", *J. Acoust. Soc. Am.*, vol. 95, no. 2, pp. 770-782, Feb. 1994.
- [2] J.F. Böhme, "Statistical array signal processing of measured sonar and seismic data", in *Proc. SPIE 2563 Advanced Signal Processing Algorithms*, San Diego, July 1995.

$$\begin{aligned}
\mathcal{M}_1 : \vartheta_1 &= (\text{sr}, \text{sd})' \\
\mathcal{M}_2 : \vartheta_2 &= (\text{sr}, \text{sd}, \text{tilt})' \\
\mathcal{M}_3 : \vartheta_3 &= (\text{sr}, \text{sd}, \text{tilt}, \text{bathy}(2))' \\
\mathcal{M}_4 : \vartheta_4 &= (\text{sr}, \text{sd}, \text{tilt}, \text{bathy}(2), c_{0,5,20,50})' \\
\mathcal{M}_5 : \vartheta_5 &= (\text{sr}, \text{sd}, \text{tilt}, \text{bathy}(2), c_{0,5,20,50}, \text{rd}, c_b, \text{sed.att.})'
\end{aligned}$$

Figure 2. Partitioning of ϑ into subsequent model orders $\mathcal{M}_1, \mathcal{M}_2, \dots$ of increasing complexity. Parameters: source range, source depth, two parameters for bathymetry, array tilt, velocity profile, receiver depth of deepest hydrophone in array, bulk velocity in sediment, sediment attenuation

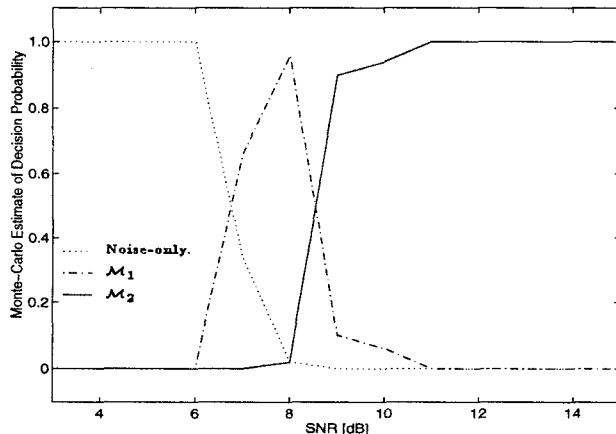


Figure 3. Results of 50 Monte Carlo simulations at each SNR. The figures show the calculated test decision behaviour for false alarm rate $\alpha = 5\%$. This shows the computed frequencies of the test outcomes for testing \mathcal{M}_1 against \mathcal{M}_2 .

- [3] R.H. Shumway, "Replicated time-series regression: An approach to signal estimation and detection", in *Handbook of Statistics*, Vol. 3, D.R. Brillinger and P.R. Krishnaiah, Eds., pp. 383-408. Elsevier Science Publishers B.V., 1983.
- [4] F.B. Jensen and M.C. Ferla, *SNAP: The SACLAN-T-CEN Normal-Mode Acoustic Propagation Model*, SACLAN-T ASW Research Centre, La Spezia/Italy, Jan. 1979.
- [5] L. Ljung, *System Identification, Theory for the User*, Information and System Sciences Series. Prentice-Hall, 1987.
- [6] D.J. Thomson, "Spectrum estimation and harmonic analysis", *Proceedings of the IEEE*, vol. 70, no. 9, pp. 1055-1096, Sept. 1982.
- [7] D.B. Percival and A.T. Walden, *Spectral Analysis for Physical Applications, Multitaper and Conventional Univariate Techniques*, Cambridge University Press, 1993.
- [8] C.F. Mecklenbräuker and P. Gerstoft, "Hypothesis testing for acoustic environmental models using likelihood ratio", in *Proc. 3rd Europ. Conf. Underwater Acoustics*, Papadakis, Ed., Heraklion, Crete, June 1996, European Commission, pp. 465-470, Crete University Press.
- [9] D.F. Gingras, "North Elba Sea Trial Summary", Tech. Rep., NATO SACLAN-T-CEN, La Spezia, Aug. 1994, report and acoustic dataset are published at the Signal Processing Information Base at Rice University on the World Wide Web, <http://spib.rice.edu/>.
- [10] D.F. Gingras and P. Gerstoft, "Inversion for geometric and geoaoustic parameters in shallow water: experimental results", *J. Acoust. Soc. Am.*, vol. 97, pp. 3589-3598, 1995.

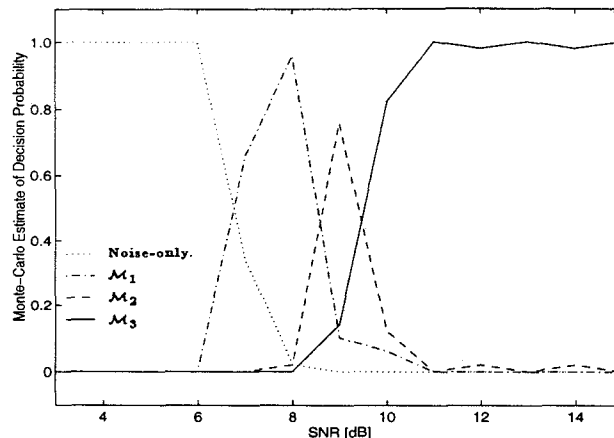


Figure 4. Results of 50 Monte Carlo simulations at each SNR. The figures show the calculated test decision behaviour for false alarm rate $\alpha = 5\%$. Result of the sequential test upto \mathcal{M}_3 .

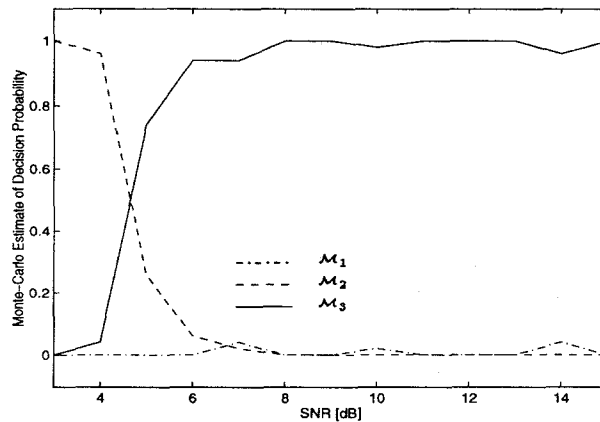


Figure 5. Results of 50 Monte Carlo simulations at each SNR. The figures show the calculated test decision behaviour for AIC upto \mathcal{M}_3 .

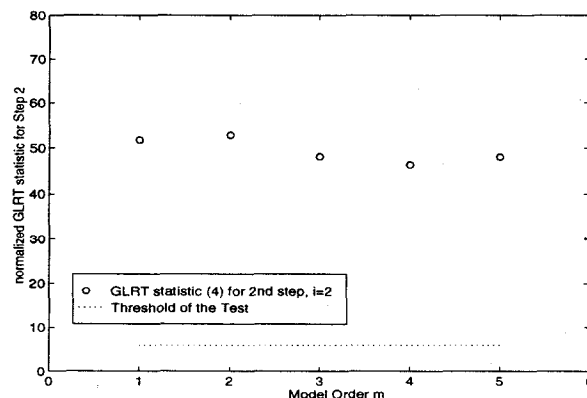


Figure 6. Plot of the $T_{2,m}$ test statistic for the N-Elba data set, cf. [8].