SPARSE BAYESIAN LEARNING FOR DOA ESTIMATION USING CO-PRIME AND NESTED ARRAYS

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ABSTRACT

Sparse Bayesian learning (SBL) has been used to obtain source direction-of-arrivals (DoAs) from uniform linear array (ULA) data. The maximum number of sources that can be resolved using a ULA is limited by the number of sensors in the array. It is known that sparse linear arrays such as co-prime and nested arrays can resolve more sources than the number of sensors. In this paper we demonstrate this using SBL. We compute the mean squared error in source power estimation as various parameters are varied.

***Index Terms*—** Sparse Bayesian learning, co-prime array, nested array, DoA estimation, compressed sensing

1. INTRODUCTION

Sparse Bayesian learning (SBL) [1, 2] is a compressive sensing technique that can find sparse solutions to an underdetermined linear problem. In the context of direction-of-arrival (DoA) estimation using uniform linear array (ULA) sensor data it has been applied to resolve nearby sources [3, 4]. A limitation of ULA data is that the maximum number of sources that can be resolved is limited by the number of sensors in the array. A Cramér-Rao based theoretical analysis of SBL in this context has been performed in [5, 6, 7].

Recently various sparse array geometries have been proposed such as Nested arrays [8] and Co-prime arrays [9, 10, 11, 12, 13] which can resolve more sources than the number of sensors. This has been demonstrated with MUSIC [14, 15, 16] and SBL [17] using the co-array covariance. Theoretical results justifying these claims have also appeared in the literature [16, 18].

In this paper we use SBL to directly process observation vectors from nested and co-prime arrays. It is shown using simulations that SBL can identify more sources than the number of sensors. A mean squared error comparison of ULA, nested and co-prime arrays, all having same number of sensors, is performed with respect to parameters such as number of snapshots, number of sources and signal-to-noise ratio.

This paper is organized as follows: Section 2 gives a brief overview of the co-prime and nested arrays. The SBL algorithm is summarized in Section 3 and a pseudocode is provided for implementation. Simulations are performed in Section 4 to study performance of SBL with respect to various parameters. Conclusions are discussed in Section 5.

1.1. Signal Model

In the sparse signal representation framework, the lth observation snapshot $\mathbf{y}_l$ recorded by an array with N sensors due to impinging plane waves is given by $\mathbf{y}_l = \mathbf{A}\mathbf{x}_l + \mathbf{n}_l$, where $\mathbf{y}_l \in \mathbb{C}^N$, $\mathbf{x}_l \in \mathbb{C}^M$, M is the number of grid points in which the angle space $[-90, 90]$ is divided, and $\mathbf{n}_l$ is the additive complex Gaussian noise. The sparse vector $\mathbf{x}_l$ has at most $K \ll M$ non-zero entries corresponding to the complex amplitudes of the waves. The objective is to find the unknown vector $\mathbf{x}_l$ given the observations $\mathbf{y}_l$ and the dictionary $\mathbf{A}$. Typically multiple (L) snapshots are used $\mathbf{Y} = [\mathbf{y}_1, \dots, \mathbf{y}_L]$.

The columns of dictionary $\mathbf{A}$ are composed of the steering vectors corresponding to the M discrete angles $\{\theta_1, \dots, \theta_M\}$. For a narrow-band signal of wavelength λ and sensor locations given by $\{d_1, \dots, d_N\}$, the mth column is

$$\mathbf{a}_m = [1, e^{j2\pi\frac{d_1}{\lambda}\sin(\theta_m)}, \dots, e^{j2\pi\frac{d_N}{\lambda}\sin(\theta_m)}]^T. \tag{1}$$

2. CO-PRIME AND NESTED ARRAYS

A ULA consists of uniformly spaced sensors with locations $d_n = (n-1)d, d = 1, \dots, N$ where d is the uniform spacing. A co-prime array consists of two ULAs with N_1 and N_2 sensors such that N_1 and N_2 are co-prime (i.e. their greatest common divisor is 1). Also let $N_1 > N_2$ without loss of generality. If d is the fundamental spacing, location of sensors in a co-prime array is given by the following set

$$\begin{aligned} s =& \{0, dN_2, \dots, (N_1-1)dN_2\} \\ &\cup \{dN_1, 2dN_1, \dots, (2N_2-1)dN_1\}. \end{aligned} \tag{2}$$

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A co-prime array has a total of $(N_1 + 2N_2 - 1)$ sensors.

A nested array also consists of two ULAs, the dense ULA portion with N_1 sensors (spacing d), and the second ULA portion with N_2 sensors (spacing $(N_1+1)d$). The set of sensor locations for a nested array is given by

$$\begin{aligned} s =& \{d, 2d, \ldots, N_1 d\} \\ & \cup \{(N_1+1)d, 2(N_1+1)d, \ldots, (N_1+1)dN_2\}. \end{aligned} \tag{3}$$

A nested array consists of a total of $(N_1 + N_2)$ sensors. For a detailed discussion on co-prime and nested arrays refer [8, 9]. An advantage of co-prime and nested arrays is that they can detect more sources than the number of sensors. Typically the spacing d is chosen to be $d = \frac{\lambda}{2}$.

3. REVIEW OF SPARSE BAYESIAN LEARNING

The multi-snapshot signal model is given by

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{N}, \tag{4}$$

where the noise $\mathbf{N} = [\mathbf{n}_1, \ldots, \mathbf{n}_L]$ with $\mathbf{n}_l \sim \mathcal{CN}(\mathbf{n}_l; \mathbf{0}, \sigma^2\mathbf{I})$, $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_L]$ is the matrix of sparse weights with all the columns sharing the same sparsity. The observations are assumed to be independent across the snapshots. The multi-snapshot likelihood function is written as

$$p(\mathbf{Y}|\mathbf{X}) = \prod_{l=1}^{L} p(\mathbf{y}|\mathbf{x}) = \prod_{l=1}^{L} \mathcal{CN}(\mathbf{y}_l; \mathbf{A}\mathbf{x}_l, \sigma^2\mathbf{I}). \tag{5}$$

Prior: In SBL, $\mathbf{x}$ is treated as a zero mean complex Gaussian random vector with unknown diagonal covariance $\mathbf{\Gamma} = \text{diag}(\gamma_1 \ldots \gamma_M) = \text{diag}(\boldsymbol{\gamma})$. The prior model is given by

$$p(\mathbf{X}) = \prod_{l=1}^{L} p(\mathbf{x}_l) = \prod_{l=1}^{L} \mathcal{CN}(\mathbf{x}_l; \mathbf{0}, \mathbf{\Gamma}). \tag{6}$$

Evidence: For Gaussian prior and likelihood, the evidence $p(\mathbf{Y})$ is Gaussian and given by

$$p(\mathbf{Y}) = \int p(\mathbf{X}) p(\mathbf{Y}|\mathbf{X}) d\mathbf{X} = \prod_{l=1}^{L} \mathcal{CN}(\mathbf{y}_l; \mathbf{0}, \mathbf{\Sigma}_\mathbf{y}), \tag{7}$$

where $\mathbf{\Sigma}_\mathbf{y} = \sigma^2\mathbf{I} + \mathbf{A}\mathbf{\Gamma}\mathbf{A}^H$. The SBL approach is to estimate the diagonal entries of $\mathbf{\Gamma}$ by maximizing the (log) evidence

$$(\hat{\gamma}_1 \ldots \hat{\gamma}_M) = \arg\max_{\boldsymbol{\gamma}} \left\{ -\sum_{l=1}^{L} \mathbf{y}_l^H \mathbf{\Sigma}_\mathbf{y}^{-1} \mathbf{y}_l - L\log|\mathbf{\Sigma}_\mathbf{y}| \right\}.$$

Differentiating and equating the derivatives to zero gives the fixed point update rule [1, 2, 4]

$$\begin{aligned} \gamma_m^{\text{new}} &= \gamma_m^{\text{old}} \frac{1}{L} \frac{||\mathbf{Y}^H \mathbf{\Sigma}_\mathbf{y}^{-1} \mathbf{a}_m||_2^2}{\mathbf{a}_m^H \mathbf{\Sigma}_\mathbf{y}^{-1} \mathbf{a}_m} && (8) \\ &= \gamma_m^{\text{old}} \frac{\text{Tr}[\mathbf{S}_\mathbf{y} \mathbf{\Sigma}_\mathbf{y}^{-1} \mathbf{a}_m \mathbf{a}_m^H \mathbf{\Sigma}_\mathbf{y}^{-1}]}{\mathbf{a}_m^H \mathbf{\Sigma}_\mathbf{y}^{-1} \mathbf{a}_m} && (9) \end{aligned}$$

where $\mathbf{S}_\mathbf{y} = \frac{1}{L}\mathbf{Y}\mathbf{Y}^H$ is the sample covariance matrix (SCM) and Tr$[\cdot]$ denotes the trace operator for a matrix.

Using stochastic likelihood we can formulate a noise variance update rule [4]. But this noise estimate is not valid for more sources than sensors $(K \geq N)$. Hence to keep simulations simple, in this paper it is assumed that the noise is known exactly and we focus on estimating γ. The pseudocode of the SBL algorithm is given in Algorithm 1.

Algorithm 1 Sparse Bayesian Learning

```
Parameters: ε = 10^{-3}, N_t = 1200
Input: Y, A, σ^2
Initialization: γ_m^old = 1, ∀m
for i = 1 to N_t
    Compute: Σ_y = σ^2 I_N + A Γ^old A^H
    γ_m^new update ∀m using (9)
    If ||γ^new − γ^old||_1 / ||γ^old||_1 < ε, break
    γ^old = γ^new, Γ^old = diag(γ^old)
end
```

To resolve more sources than the number of sensors, SBL Algorithm 1 can directly be applied to observations from co-prime and nested arrays. This is possibly because from (9) the update rule depends on the sample covariance matrix $\mathbf{S}_\mathbf{y}$ which has more diversity than directly using the observations $\mathbf{Y}$. The dimensions of the covariance matrices required by SBL are $O((N_1 + N_2) \times (N_1 + N_2))$.

By comparison, MUSIC based on direct SCM can only find up to $O(N_1 + N_2)$ sources as there are at most $O(N_1 + N_2)$ eigenvalues. It has been demonstrated that MUSIC [14] with co-array based covariance can resolve more sources than the number of sensors. This requires construction of a higher dimensional covariance matrix of size $O(N_1N_2 \times N_1N_2)$ from the smaller $O((N_1 + N_2) \times (N_1 + N_2))$ direct SCM. SBL does not rely on eigendecomposition and is thus able to extract higher number of sources from the smaller, direct SCM itself. Alternately, a covariance based LASSO [19] could also recover more sources but the computational costs would be higher than those of SBL (see [4], Fig.3).

4. SIMULATIONS

4.1. Gram matrices

In simulations we apply SBL to measurements obtained from three array geometries: uniform linear array (ULA), co-prime array, and nested array. All three arrays have the same number of sensors, $N = 9$. To achieve this we construct the co-prime array with $N_1 = 4, N_2 = 3$ and nested array with $N_1 = 5, N_2 = 4$. The sensor positions for the three cases are given in Table 1. We use $d = \frac{\lambda}{2}$ where λ is the signal wavelength.

The corresponding Gram matrices $|\mathbf{A}^H\mathbf{A}|$ for each of the arrays is shown in Figure 1. The angle space $[-90, 90]$ is discretized using a grid of size $M = 44$ giving a resolution of $\Delta\theta = \frac{180}{43} \sim 4^\circ$. We note that due to the specific sensor arrangement, co-prime and nested arrays have larger aperture than ULA for the same number of sensors. In examples the ULA, co-prime and nested arrays have apertures of $8d$, $20d$ and $23d$ respectively. Though co-prime and nested arrays have much larger apertures than ULA, they do not cause aliasing (for the same number of sensors, the aperture of ULA cannot be further increased without causing aliasing).

Array	Sensor positions	Aperture
ULA	$\{0, 1, \ldots, 8\}d$	$8d$
Co-prime	$\{0, 3, 6, 9, 4, 8, 12, 16, 20\}d$	$20d$
Nested	$\{1, 2, 3, 4, 5, 6, 12, 18, 24\}d$	$23d$

Table 1: The three arrays used in simulations along with the location of the sensors and the array aperture.

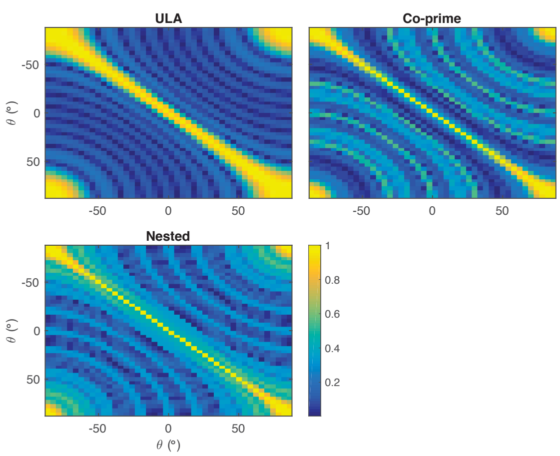


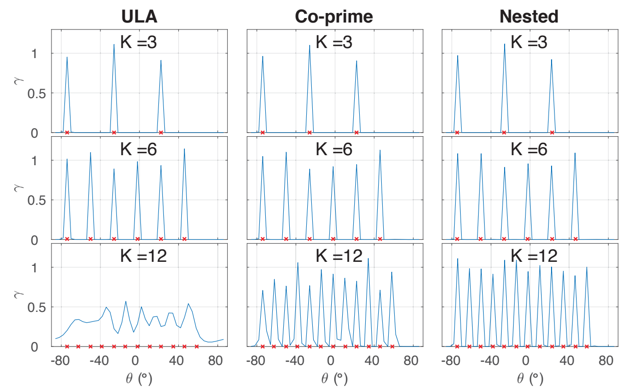
Fig. 1: Gram matrices $|\mathbf{A}^H\mathbf{A}|$ for the three array configurations: ULA, Co-prime, and Nested. All the arrays have same number of sensors $N = 9$ with grid size $M = 44$.

4.2. Resolving more sources than sensors

The SBL algorithm in Section 3 is applied for DoA estimation. We simulate observations assuming multiple sources with equal amplitudes. The angle resolution is $\Delta\theta \sim 4^\circ$ using a grid of size $M = 44$. The sources are always assumed to be on one of the grid points.

The estimates $\hat{\gamma}$ from a typical run of SBL are shown in Figure 2a for the three arrays and three different sparsity values, i.e. $K = 3, 6, 12$ sources. Data is processed using $L = 100$ snapshots with an SNR of 20 dB. The true value of γ is 1 at its non-zero location. The co-prime and nested arrays can resolve all the DoAs even when more sources are present than the number of sensors $(K = 12, N = 9)$, while

(a) Plot of γ



(b) Plot of γ_{mean}

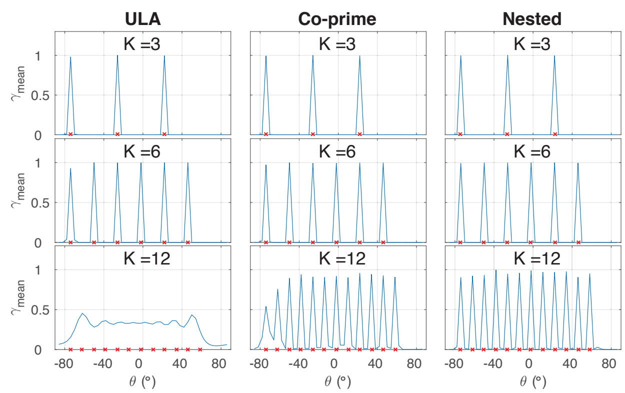


Fig. 2: (a) A typical γ at convergence for ULA, co-prime, and nested arrays. All arrays have $N = 9$ sensors. $L = 100$ and SNR is 20 dB. (b) Mean γ over 1000 Monte Carlo runs.

the solution for ULA is not sparse. The mean of γ over 1000 Monte Carlo (MC) runs is shown in Figure 2b.

4.3. Mean squared error

The performance of SBL algorithm processing measurements from different array types can be quantified using the mean squared error (MSE). The MSE is calculated as

$$\text{MSE}(\gamma) = \frac{1}{N_{\text{sim}}} \frac{1}{M} \sum_{i=1}^{N_{\text{sim}}} \sum_{m=1}^{M} (\gamma_{i,m} - \hat{\gamma}_{i,m})^2 \tag{10}$$

where γ_i and $\hat{\gamma}_i$ are the true and the estimated γ for the ith MC simulation run, N_{sim} is the number of MC runs. When expressed in log scale we have $\text{MSE(dB)} = 10\log_{10}(\text{MSE})$.

Figure 3 plots the above MSE of γ as a function of number of snapshots for $K = 6, 9, 12$ sources. The number of sensors is $N = 9$ and noise is added at an SNR of 20 dB. The MSE is highest for ULA in all the cases. For $K = 6, 9$ both co-prime and nested array have similar error. The nested array has the least error for $K = 12$.

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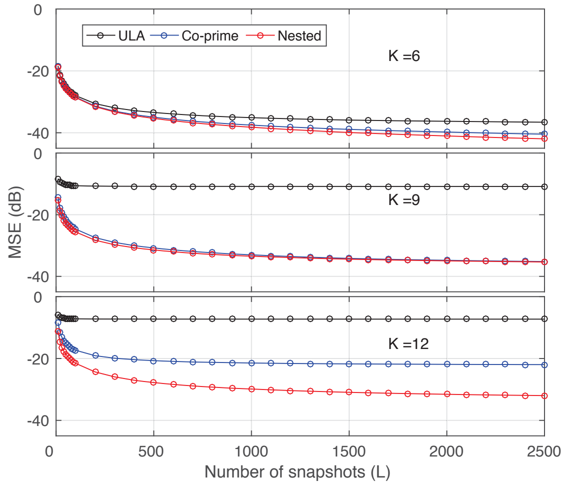

Fig. 3: MSE (γ) vs L for ULA, co-prime and nested arrays. All arrays have $N = 9$ sensors and SNR is set to 20 dB.

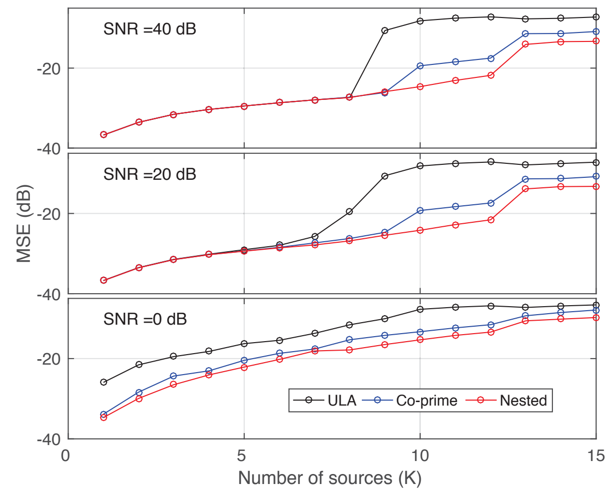

Fig. 5: MSE (γ) vs K for ULA, co-prime and nested arrays. All arrays have $N = 9$ sensors and $L = 100$ snapshots.

Figure 4 shows the MSE as the SNR is varied in the simulation. For $K = 6$ all three arrays have very similar MSE performance. On increasing the number of sources to $K = 9$ the ULA MSE significantly degrades as it cannot resolve all the sources. Further increasing $K = 12$ gives lowest MSE for nested array, followed by co-prime array and ULA. This is also confirmed by the plots of γ_{mean} in Figure 2b.

We also compute MSE as a function of the number of sources (K). For this simulation the number of snapshots $L = 100$ is fixed and SNR is selected from $0, 20, 40$ dB as seen in Figure 5. As observed from previous simulations, the increasing order of MSE is nested $<$ co-prime $<$ ULA.

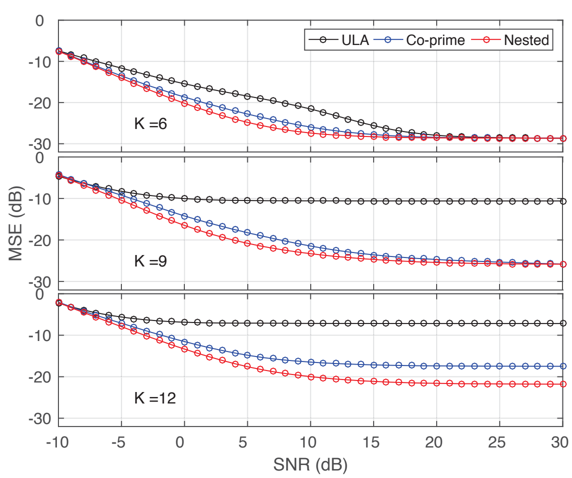

Fig. 4: MSE (γ) vs SNR for ULA, co-prime and nested arrays. All arrays have $N = 9$ sensors and $L = 100$ snapshots.

We now consider the root MSE (RMSE) of DoA estimation. The DoAs are computed as the location of the strongest K peaks in γ. The Cramér-Rao bound (CRB) for DoA estimation for the case $K > N$ is given in [16], Eq.(19). The grid separation is $\Delta\theta = 0.5°$ giving $M = 359$. SNR is 0 dB, $K = 12$, and 100 MC runs are performed. Figure 6 plots the DoA RMSE vs snapshots (L) for co-prime and nested arrays. The corresponding CRB is also shown. [1]

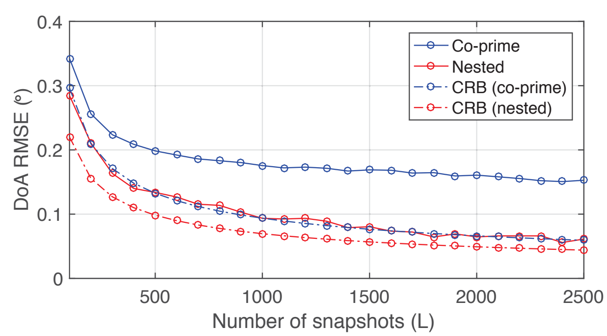

Fig. 6: DoA RMSE/CRB vs L for co-prime and nested arrays. Number of sensors $N = 9$, $K = 12$ sources, and SNR 0 dB.

5. CONCLUSIONS

SBL is able to resolve more sources than number of sensors when measurements from sparse linear arrays such as co-prime and nested arrays are processed. The mean squared error performance was studied by varying the parameters: number of snapshots, SNR, and number of sources. For the same number of sensors, the MSE is lowest for nested array, followed by co-prime array, and ULA.

[1]We are thankful to Y. Qin [17] for providing CRB MATLAB code.

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