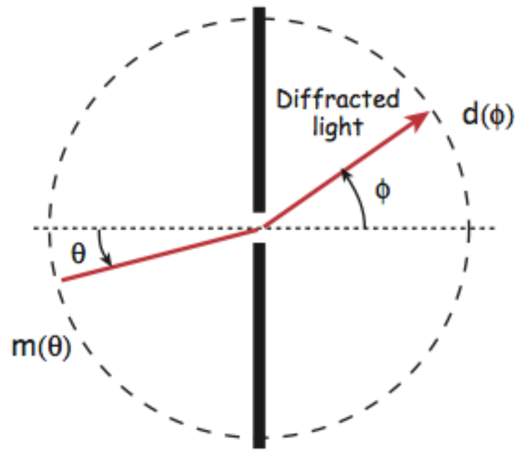
Singular value decomposition

When light passes through a thin slit it is diffracted. The distribution of light intensity as a function of incidence angle θ is represented by $m(\theta)$. After passing through the slit it becomes $d(\phi)$, where ϕ is the angle as defined in the figure. $m(\theta)$ and $d(\phi)$ are related by the expression

$$d(\phi) = \int_{-\pi/2}^{\pi/2} (\cos(\phi) + \cos(\theta))^2 \left(\frac{\sin(\pi(\sin(\phi) + \sin(\theta)))}{\pi(\sin(\phi) + \sin(\theta))} \right)^2 m(\theta) d\theta \qquad (1)$$



Here the data are measured intensity $d(\phi)$ at N equal intervals, $d(\phi_i) = d_i, (i = 1, \ldots, N)$ where $-\frac{\pi}{2} <= \phi_i <= \frac{\pi}{2}$, and the model is the incident intensity $m(\theta)$ discretized over the same angular intervals, $m(\theta_j) = m_j, (j = 1, \ldots, N)$, which leads to a discrete linear system of $N \times N$ equations, $\mathbf{d} = G\mathbf{m}$, where

$$G_{i,j} = \Delta\phi(\cos(\phi_i) + \cos(\theta_j))^2 \left(\frac{\sin(\pi(\sin(\phi_i) + \sin(\theta_j)))}{\pi(\sin(\phi_i) + \sin(\theta_j))} \right)^2 \qquad (2)$$

This results in a severely ill-posed inverse problem. MATLAB routine `[G,m,d]=shaw(20)` computes the G matrix along with a sample model and data for this problem with $N = 20$.

1. Calculate the singular values of the G matrix. Plot singular values s_i as a function of index i on a semi-log plot. What do you notice about the shape ?

[You may find routine svd helpful here `[U,S,V]=svd(G)`. Note that eigenvalues are along the diagonal of matrix S (`diag(S)`). and a semilog plot is achieved with routine `semilogy`.]

2. Use MATLAB commands to get the rank and condition number of G. What do these terms mean ? Can you see how they are affect in the eigenvalue spectrum ?

3. Plot the eigenvector corresponding to the smallest non zero eigenvalue (i.e. s_p, where p=rank of G). What do you notice about its shape ?

4. Plot the eigenvector corresponding to the largest and 5th largest eigenvalue. Look at these

together with your previous plot. How does the shape changes with eigenvalue ?

5. Generate an input spike model **m** with zeros everywhere and the 10th element equal to 1. Plot this model, calculate the corresponding data using **d** = G**m** and plot this. [You may find routine `zeros(20,1)` useful here.]

6. Apply the generalized inverse $G^{\dagger}\mathbf{d} = V\ S^{-1}U^{T}\mathbf{d}$ to obtain solutions for different values of p. Choose the highest p and lowest p and compare the models. How good is the recovery? [Note that `p=10; Up=U(:,1:p);` will set `Up` to the matrix containing the first `p` columns of `U`.]

7. Now add 20 realizations of normal random noise with zero mean and standard deviation $\sigma = 10^{-6}$ to the data vector. Plot the noisy data and the no noise data on the same figure. Do you see much difference ? [You may find a command of this form useful here

```
dn=spikedata+1.0e-6*randn(size(spikedata));]
```

8. Calculate the solution for the noisy data, using the full rank of the generalized inverse, i.e. `p=rank(G)`. Plot the solution. How does it compare to the true solution? What has happened?

9. Calculate a solution by truncating the SVD to a smaller number of singular values and find a value for p which you think best recovers the true spike model. By reducing p what have you managed to do to the inverse problem ? What has been sacrificed to achieve this.